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Koszul Duality and Deformation Theory

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This thesis is submitted for the degree of

Doctor of Philosophy

May 2021

Abstract

of the thesis

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September 2020

Revised May 2021

This thesis answers various questions related to Koszul duality and deformation theory. We begin by giving a general treatment of deformation theory from the point of view of homotopical algebra following Hinich, Manetti and Pridham. In particular, we show that any deformation functor in characteristic zero is controlled by a certain differential graded Lie algebra defined up to homotopy, and also formulate a noncommutative analogue of this result valid in any characteristic.

In the next part of this thesis, we introduce a notion of left homotopy for Maurer–Cartan elements in L_∞-algebras and A_∞-algebras, and show that it corresponds to gauge equivalence in the differential graded case. From this we deduce a short formula for gauge equivalence, and provide an entirely homotopical proof to Schlessinger–Stasheff's theorem. As an application, we answer a question of T. Voronov, proving a non-abelian Poincaré lemma for differential forms taking values in an L_∞-algebra.

In the final part of this thesis, we generalize previous formulations of Koszul duality for associative algebras by Keller–Lefèvre and Positselski. For any dg algebra A we construct a model category structure on dg A-modules such that the corresponding homotopy category is compactly generated by dg A-modules that are finitely generated and free over A (disregarding the differential). We prove that this model category is Quillen equivalent to the category of comodules over a certain, possibly nonconilpotent differential graded coalgebra, a so-called extended bar construction of A.

Acknowledgements

To my supervisor Andrey Lazarev – for being an endless source of ideas and inspiration, for generously sharing his knowledge and time, for numerous conversations that brought fresh perspectives, for his detailed attention to my work, for his patience, gentle encouragement and good humour, and for so much more.

To my former co-supervisor Chris Braun – for fielding my silly questions with a cheery enthusiasm that made every meeting something to look forward to.

To Yemon Choi, Samuel Colvin, Edwin Kutas, Nadia Mazza, David Pauksztello, and Dirk Zeindler – for their words of reassurance in those toughest moments.

To friends and rivals from the Go clubs of Lancaster and Manchester – for the good games, for the good company, every week over the past four years.

To my mum, to whom this thesis is dedicated – for her quiet and unwavering support every step of the way.

Thank you.

Declaration

This thesis is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in Section 1.2.

This thesis has not been submitted in substantially the same form for the award of any other degree or qualification.

This thesis does not exceed the permitted maximum of 80,000 words.

Contents

CHAPTER 1

Introduction

It has long been observed that any reasonable deformation theory is "controlled" by a differential graded (dg) Lie algebra, at least in characteristic zero. This central principle of deformation theory, first recorded by Deligne in a letter to Millson [GM88], has been demonstrated many times, starting from the work of Nijenhuis and Richardson in the 1960s [NR64, NR67]. Recently this principle has been formalized by [Pri10] using simplicial techniques and by [Lur] in the framework of infinity categories. In the first part of this thesis, we show that this result can be proved in a much more elementary way, with Koszul duality playing an essential role in its proof.

The standard approach to deformation theory is as follows. Suppose that O is an object of a certain category that one wants to deform and which is defined, in some sense, over some ground field k. We will not attempt to axiomatize this situation but a good example to keep in mind is an associative algebra over k. Then, a deformation of O over a finite dimensional (Artinian) local ring K is an object O_K defined, in the same vague sense, over K and such that its 'reduction' modulo the maximal ideal I of K is isomorphic to O. Two deformations O_K and O'_K are *equivalent* if O_K and O'_K are isomorphic via an isomorphism that is the identity modulo I. Thus, we have a functor Def associating to a local Artinian ring K the set of equivalence classes of deformations of O over K. The fundamental problem of deformation theory is finding a 'universal' ring K_u and the corresponding universal deformation of O over K_u, i.e. an element in $\mathrm{Def}(K_u)$ so that any other deformation of O over K is induced by a unique map $K_u \to K$.

It has been understood for a long time that one can only expect the deformation functor to be *pro-representable*, in other words we might as well extend the category on which Def is defined to include projective limits of local Artinian k-algebras (we will call them local pseudocompact k-algebras). Furthermore, it makes sense to attempt to characterize functors on the category of local Artinian (or pseudocompact) rings that deformation functors satisfy in concrete examples and then investigate whether these characteristic properties ensure (pro)representability. Note that a representable functor preserves arbitrary limits; moreover under some mild conditions on the category of set-theoretical nature, any functor preserving limits is representable (the so-called Freyd's adjoint functor theorem, [Fre64]). However, a deformation functor may not preserve limits; indeed infinitesimal automorphisms often present an obstruction to such a preservation, cf. [Sch68, Remark 2.15] for an explanation of this point. On the other hand, we often have that it preserves arbitrary products and that the natural map of sets

$$\mathrm{Def}(B \times_A C) \to \mathrm{Def}(B) \times_{\mathrm{Def}(A)} \mathrm{Def}(C) \tag{1.0.1}$$

is *surjective* (if it is bijective this would imply that Def preserves arbitrary colimits). Additionally, it usually makes sense to impose the *normalization condition*: $\mathrm{Def}(k)$ is a one-point set. Together with another mild condition on infinitesimal deformations, these imply that Def has a *hull*, a certain weakening of the property of being representable, [Sch68, Theorem 2.11].

In order to obtain a decisive general result, it is necessary to extend the category of local pseudocompact algebras to that of local *differential graded* pseudocompact algebras. The advantage of the latter is that it has the structure of a *model category* and, in particular one can form its *homotopy category*. This model category was constructed in a seminal paper of Hinich [Hin01] and an extended deformation functor was considered in [Man02, Mer00]. The latter papers, however, did not make full use of the strength of the model structure on local pseudocompact dg algebras.

So, we now have a set-valued functor defined on the category of local pseudocompact dg algebras. It is a deformation functor if it is normalized, preserves arbitrary products, has an appropriate analogue of (1.0.1) and, crucially, is *homotopy invariant*, so that it descends to a functor on the homotopy category of local pseudocompact dg algebras. In the commutative case and when k has characteristic zero, we will show that, under these conditions the functor is representable in the homotopy category and there is a certain dg Lie algebra, defined up to a quasi-isomorphism "controlling" it. In the associative case we will similarly show that, under these conditions the functor is representable in the homotopy category and there is a certain dg *associative* algebra, defined up to a quasi-isomorphism "controlling" it; this will be valid in any characteristic.

In Chapter 3, we explain how Deligne's principle can be made into a rigorous theorem. The approach that we take relies on three fundamental results that are important and interesting in their own right:

(1) Koszul duality between dg Lie algebras and cocommutative dg conilpotent coalgebras as formulated by Hinich [Hin01], as well as its associative variant [Pos11];
(2) A theorem of Schlessinger and Stasheff [SS];
(3) A model category version of Brown's representability theorem [Bro62] due to [Jar11].

The final two chapters of this thesis are concerned with generalizations of the Schlessinger–Stasheff theorem and of Koszul duality respectively. In the next part of this introduction, we give an overview of our work in these two directions.

The Schlessinger–Stasheff theorem. A Maurer–Cartan element in a dg Lie algebra V is a degree 1 element $\xi \in V$ satisfying $d\xi + \frac{1}{2}[\xi, \xi] = 0$. It is important to understand homotopies between Maurer–Cartan elements; for example, deformation problems are governed by Maurer–Cartan elements up to an appropriate notion of homotopy [SS, Man99]. Thus many different notions of homotopy have been studied for Maurer–Cartan elements in dg Lie algebras and, more generally, in L_∞-algebras; see [DP16] for an up-to-date and extensive survey.

The Schlessinger–Stasheff theorem [SS] states that two Maurer–Cartan elements in a pronilpotent dg Lie algebra are Sullivan homotopic (called Quillen homotopic in [DP16]) if and only if they are gauge equivalent. In Chapter 4, we provide an entirely homotopical proof of this result, and extend to it to L_∞-algebras and A_∞-algebras under certain completeness conditions. To do this, we introduce a new homotopy relation for Maurer–Cartan elements in complete L_∞-algebras and A_∞-algebras. Maurer–Cartan elements are interpreted as morphisms of commutative differential graded algebras (cdgas); this is reviewed in Section 4.1 along with other relevant background on L_∞-algebras and A_∞-algebras. Two Maurer–Cartan elements are then defined to be left homotopic if they are left homotopic between morphisms in the category of cdgas, equipped with the model category structure of [Hin97].

As motivation for our definition, in Section 4.2 we show that there is a model structure on the category of complete dg Lie algebras, namely that of [LM15], in which gauge equivalence coincides with left homotopy, a result also obtained by [RN18]. The results in this section should be considered Koszul dual to the approach taken in the rest of the paper, where we choose to work in the setting of cdgas in order for results to immediately generalize to the setting of L_∞-algebras. There are also close parallels between the approach in Section 4.2 and the recent papers [BM13b, BFMT18], in which it is shown that gauge equivalence coincides with left homotopy for a larger class of dg Lie algebras with a different model structure. However, their result only holds in the generality of dg Lie algebras, and the method used does not seem to easily generalize to L_∞-algebras.

As an application, we answer a question posed by Voronov in [Vor12], and prove a version of the Poincaré lemma for differential forms taking values in an L_∞-algebra. The notion of homotopy that we use is different from that used by Voronov, who considers homotopies for arbitrary odd elements in dg Lie superalgebras, but in the specific case of Maurer–Cartan elements in complete dglas, the notions will coincide.

Koszul duality. Koszul duality is a phenomenon occurring widely throughout algebra and geometry, going back to Quillen's work [Qui69] on rational homotopy theory, where it manifests as a duality between Quillen's Lie model and Sullivan's commutative model for a space. Another classical example is the Bernstein–Gelfand–Gelfand (BGG) correspondence [BGS96] between bounded derived categories of finitely generated modules over symmetric and exterior algebras on a generating vector space. Since then, Koszul duality has appeared in the study of operads [GK94], deformation theory [Hin01], representation theory, algebraic geometry and numerous other contexts.

A common theme in early examples of Koszul duality is the prevalence of finiteness and boundedness conditions that are essential for the dualities to hold. For example, if one tries to extend the BGG correspondence to the full derived category, this already fails for symmetric and exterior algebras on one generator. To remove these finiteness conditions, exotic derived categories were introduced in [Hin01] and further developed in [LH03] and [Pos11]. The weak equivalences are taken not to be all quasi-isomorphisms, but a strict subset of them. Following [Pos11], in this thesis we will collectively refer to these structures as being "of second kind".

The modern understanding of Koszul duality for differential graded (dg) algebras and dg modules has been formulated in [Pos11]. According to this formulation there is an adjunction between the categories of augmented dg algebras and conilpotent dg coalgebras, given by bar and cobar constructions, which becomes a Quillen equivalence under certain model category structures. The conilpotent dg coalgebra associated to an augmented dg algebra by this equivalence is called its Koszul dual; similarly the augmented dg algebra associated to a conilpotent dg coalgebra is called its Koszul dual. There is also a Quillen equivalence between the corresponding categories of dg modules and dg comodules. A variant of this correspondence exists for non-augmented dg algebras and their modules.

A salient feature of this theory is that the model category structures on the Koszul dual side (both for coalgebras and their comodules) are of the "second kind": the weak equivalences are not created in the underlying chain complexes but are of a more subtle nature (so-called *filtered quasi-isomorphisms*).

The module-comodule Koszul duality is the easiest one to prove (though still quite nontrivial), essentially because of its linear character: this is a duality between stable model categories whose homotopy categories are triangulated. There are two symmetric versions of it:

(1) the duality between modules over a dg algebra and dg comodules over its Koszul dual conilpotent dg coalgebra and
(2) the duality between comodules over a conilpotent dg coalgebra and dg modules over its Koszul dual dg algebra.

What happens if one drops the condition of conilpotency on the coalgebra side? The model structure on the category of comodules does not depend on the conilpotency assumption, [Pos11, Theorem 8.2]. Furthermore, Positselski proves ([Pos11, Theorem 6.7]) that there is a Koszul duality between dg comodules over a possibly nonconilpotent dg coalgebra and modules over its Koszul dual dg algebra. However, this time *both* model structures are of the second kind: the weak equivalences on dg modules are not merely quasi-isomorphisms. If the coalgebra happens to be conilpotent, then the duality specialises to the ordinary one: the Koszul dual dg algebra becomes cofibrant and weak equivalences of dg modules over a cofibrant dg algebra are ordinary quasi-isomorphisms.

In Chapter 5 we construct a complementary version of Positselski's nonconilpotent Koszul duality as a Quillen equivalence between model categories of dg modules over a dg algebra and comodules over its "Koszul dual" dg coalgebra. The difference between our version and the standard one is two-fold: firstly, the weak equivalences between dg modules are of "second kind" (i.e. they are not created in the category of underlying complexes) and secondly, our "Koszul dual" dg coalgebra is typically much bigger than the ordinary bar construction; in particular it is *not* conilpotent in general. This extended bar construction has been considered, e.g. in a recent paper [AJ].

Perhaps the most interesting feature of this correspondence is an exotic model structure of second kind on dg modules over a dg algebra: in the case of an ordinary algebra (or, more generally, cohomologically non-positively graded dg algebra) this structure reduces to the usual one; however in general it is different. There are many competing inequivalent notions of weak equivalence of the second kind for dg modules over a dg algebra (as opposed to dg comodules where there is only one such notion); some of them support model category structures, [Bec14, Proposition 1.3.6], [Pos11, Theorem 8.3]. Our structure is generally different from those considered in the mentioned references and characterised by its compatibility with Koszul duality. It is, necessarily, compactly generated (since such is the category of dg comodules over any dg coalgebra, to which it is Quillen equivalent). This model structure is relevant to the study of various triangulated categories of geometric origin: coherent sheaves on complex analytic manifolds, cohomologically locally constant sheaves on smooth manifolds, and D-modules on smooth algebraic varieties. Its prototype is contained in the paper [Blo10] where the notion of a cohesive module over a dg algebra is introduced, which is essentially the same as a cofibrant object in our model structure.

1.1. Summary of main results

The following is a summary of the main new results from this thesis. The main results from Chapter 4, on generalizing the Schlessinger–Stasheff for strongly homotopy algebras, are:

Theorem 4.3.5. A short formula is given for gauge equivalence of Maurer–Cartan elements in complete L_∞-algebras and A_∞-algebras. This result also admits a slight generalization in a particular non-complete case; see Theorem 4.3.9.

This result is used to prove the Schlessinger–Stasheff theorem in a purely homotopical way; see Theorem 4.4.1. Our main application of this result is:

Theorem 4.5.3. Given a contractible manifold M, we show that every Maurer–Cartan differential form on M with values in an L_∞-algebra is gauge equivalent to a constant. This is a strongly homotopy generalization of the non-abelian Poincaré Lemma proven in [Vor12].

We note that [Vor12] uses different notions of homotopy and gauge equivalence; however, in the context of Maurer–Cartan elements of complete dglas, the notions coincide by the aforementioned Schlessinger–Stasheff theorem.

The main results from Chapter 5, on generalizing Koszul duality to (co)modules over a non-conilpotent "extended Koszul dual", are:

Theorem 5.2.8. A cofibrantly generated model category structure of second kind exists on the category of (right) dg modules over a dg algebra, where a morphism $f\colon M \to N$ is a fibration if it is surjective, and f is a weak equivalence if it induces a quasi-isomorphism $\mathrm{Hom}_A(K, M) \to \mathrm{Hom}_A(K, N)$ for any finitely generated A-module K that is free over V after forgetting the differential.

Theorem 5.2.11. There is a Quillen equivalence between the model category structure of Theorem 5.2.8 of dg modules over a dg algebra A, and the category of dg comodules over the extended Koszul dual of A.

1.2. Thesis outline and published work

The rest of this thesis is divided into four chapters.

Chapter 2 covers basic definitions and results that will be used throughout the thesis, split into three sections, respectively on various algebraic objects, their pseudocompact versions, and finally model categories. This chapter is included mainly for the purpose of keeping the thesis self-contained and the material is all standard, except for perhaps the notation used for various categories. A reader who is familiar with these objects may safely skip this chapter and simply refer to the Index of Notation instead.

Chapter 3 is based on the paper [GLST20b], which is joint work with Andrey Lazarev, Yunhe Sheng and Rong Tang. We first give an overview of Brown's representability theorem, followed by Maurer–Cartan moduli spaces and the Schlessinger–

Stasheff theorem, and finally Koszul duality on the level of algebras. The purpose of this chapter is to show how these ingredients can be combined to give a proof of Deligne's principle that is more elementary than those previously found in the literature. The chapter also serves as motivation and background for the rest of the thesis.

Chapter 4 is based on the paper [Gua]. We begin by recalling preliminary results on L_∞- and A_∞-algebras. Next we define left homotopy of Maurer–Cartan elements and use it to prove the main result Theorem 4.3.5. We prove combinatorial formulae for left homotopy in terms of rooted trees and give a direct proof that left homotopy coincides with gauge equivalence. Finally, we give our main application Theorem 4.5.3, generalizing the non-abelian Poincaré lemma to L_∞-algebras.

Chapter 5 is based on the paper [GL], which is joint work with Andrey Lazarev. We begin by defining the extended bar-cobar adjunction, and using it to associate "Koszul dual" dg coalgebras to dg algebras. Next we recall Positselski's model structure of second kind on comodules, and prove the main results Theorem 5.2.8 and Theorem 5.2.11. Finally these results are generalized to the curved setting.

CHAPTER 2

Preliminaries

In this chapter, we introduce the background for the rest of this thesis; in particular, we fix the notation for the various categories that we will work with in later chapters. Throughout, k denotes a field equipped with the discrete topology; this applies even if k is the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Unadorned tensor products and Homs are assumed to be over k.

2.1. Algebras and modules

2.1.1. Graded objects. By a *vector space*, we always mean a $\mathbb{Z}$-graded vector space over k, i.e. a vector space V with a direct sum decomposition $V \cong \bigoplus_{i\in\mathbb{Z}} V^i$ into vector subspaces V^i of V. A nonzero element $v \in V^i$ is said to be *homogeneous of degree* i, denoted by $|v| = i$. A nonzero linear map $f\colon V \to W$ between vector spaces is said to have *degree* j if $f(V^i) \subseteq W^{i+j}$ for all $i \in \mathbb{Z}$, denoted by $|f| = j$. Vector spaces and degree 0 maps form a category, denoted by Vect, which is symmetric monoidal under the usual tensor product. Given a graded vector space V, its *suspension* ΣV is a graded vector space with $(\Sigma V)^i = V^{i+1}$ and its *dual* V^* is a graded vector space with $(V^*)^i = (V^{-i})^*$. For ease of notation, $\Sigma(V^*)$ will simply be denoted by ΣV^*, and there is a canonical isomorphism $\Sigma^{-1}V^* \cong (\Sigma V)^*$.

An *algebra* is a monoid in the symmetric monoidal category Vect. The category of algebras is denoted by Alg. Given a vector space V, the *tensor algebra on* V is

$$T(V) := \bigoplus_{n=0}^{\infty} T^n(V), \text{ where } T^n(V) := V^{\otimes n},$$

with multiplication induced by concatenation. This is the free algebra on V, in the sense that $V \mapsto T(V)$ defines a functor $\mathsf{Vect} \to \mathsf{Alg}$ that is left adjoint to the forgetful functor $\mathsf{Alg} \to \mathsf{Vect}$ sending an algebra A to its underlying vector space:

$$\mathrm{Hom}_{\mathsf{Alg}}(T(V), A) \cong \mathrm{Hom}_{\mathsf{Vect}}(V, A).$$

A *commutative algebra* is a commutative monoid in Vect. The category of commutative algebras is denoted by CAlg. Given a vector space V, the *symmetric* *algebra on* V is

$$S(V) := \bigoplus_{n=0}^{\infty} S^n(V), \text{ where } S^n(V) := (V^{\otimes n})_{\mathbb{S}_n}.$$

Here $(V^{\otimes n})_{\mathbb{S}_n}$ denotes the coinvariants of the permutation action of the symmetric group $\mathbb{S}_n$ on $V^{\otimes n}$. This is the free commutative algebra on V. When the underlying field k is of characteristic zero, $S(V)$ may be viewed as a subspace of $T(V)$ by identifying $S^n(V)$ with the invariants $(V^{\otimes n})^{\mathbb{S}_n} \subseteq T^n(V)$ of the permutation action, via the usual map

$$S^n(V) = (V^{\otimes n})_{\mathbb{S}_n} \to (V^{\otimes n})^{\mathbb{S}_n}, \quad [w] \mapsto \sum_{\sigma \in \mathbb{S}_n} \sigma \cdot w,$$

where $[w]$ denotes the class of an element $w \in V^{\otimes n}$, with inverse

$$(V^{\otimes n})^{\mathbb{S}_n} \to S^n(V) = (V^{\otimes n})_{\mathbb{S}_n}, \quad w \mapsto \frac{1}{n!}[w]. \tag{2.1.1}$$

A *Lie algebra* is a Lie object in Vect. Given a vector space V, the *free Lie algebra on* V, denoted by $L(V)$, is the Lie subalgebra of $T(V)$ generated by V, where $T(V)$ is considered as a Lie algebra with the commutator bracket

$$[x, y] := x \otimes y - y \otimes x \text{ for all } x, y \in T(V).$$

An algebra A is *augmented* if it is equipped with a surjective algebra homomorphism $\varepsilon \colon A \to k$ called an *augmentation*. The kernel of ε, called the *augmentation ideal* of A, will be denoted by $\bar{A}$.

2.1.2. Differential graded objects. A *differential graded* (*dg*) *vector space* or *cochain complex* is a vector space V together with a linear map $d \colon V \to V$ of degree 1 such that $d^2 = 0$, called a *differential*. Note that we work with cohomological grading; one can instead work with homological grading by defining differentials to have degree -1. The category of dg vector spaces is denoted by DGVect. This is also a symmetric monoidal category, where the tensor product of two dg vector spaces V and W is their tensor product $V \otimes W$ in Vect together with the differential $d_{V\otimes W} = d_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes d_W$. Given a dg vector space V, its suspension ΣV and dual V^* are also dg vector spaces; in particular, V^* remains cohomologically graded by the choice of its grading.

Definition 2.1.1. A *differential graded algebra* is a monoid in the symmetric monoidal category DGVect, that is, a graded algebra A equipped with a differential

$d\colon A \to A$ that is a derivation:

$$d(xy) = d(x)y + (-1)^{|x|} x d(y)$$

for all homogeneous $x, y \in A$. The category of dg algebras is denoted by DGA and the category of augmented dg algebras is denoted by DGA^*.

A *commutative differential graded algebra* is a commutative monoid in DGVect. The category of commutative dg algebras is denoted by CDGA.

A *differential graded Lie algebra* is a Lie object in DGVect, that is, a graded Lie algebra L equipped with a differential $d\colon L \to L$ that is a derivation with respect to $[-,-]$:

$$d([x,y]) = [d(x), y] + (-1)^{|x|}[x, d(y)]$$

for all homogeneous $x, y \in L$. The category of dg Lie algebras is denoted by DGLA.

We now consider modules over a dg algebra A. By default, we work with *right* dg modules over dg algebras, unless stated otherwise.

Definition 2.1.2. A (*right*) *differential graded A-module* is a graded right A-module M equipped with a differential $d_M\colon M \to M$ that is compatible with the module action and the differential d of A:

$$d_M(ma) = d_M(m)a + (-1)^{|m|} m d(a),$$

for all homogeneous $m \in M$, $a \in A$. The category of (right) dg A-modules over a dg algebra A is denoted by $\mathsf{DGMod}\text{-}A$.

The dual notion of a *left differential graded A-module* can be defined conveniently by considering A^{op}, the *opposite dg algebra* of A, which has the same underlying dg vector space structure as A but multiplication given by $ab := (-1)^{|a||b|} b \cdot a$ for all $a, b \in A$, where $\cdot$ denotes the original multiplication in A. Then a left dg A-module is simply a dg A^{op}-module.

Definition 2.1.3. Let A and B be dg algebras. A *differential graded A-B-bimodule* is a right dg B-module M that is also a left dg A-module, such that the module structures commute: $a(mb) = (am)b$ for all homogeneous $a \in A$, $b \in B$, $m \in M$.

2.2. Pseudocompact algebras and modules

The purpose of this section is to give a dictionary between the language of pseudocompact algebras, used in this thesis, and coalgebras, which are commonly used in the literature.

2.2.1. Pseudocompact vector spaces. We will need a certain amount of theory of topological vector spaces, although we will be dealing with one of the simplest possible type of topological vector space: pseudocompact spaces.

DEFINITION 2.2.1. A *pseudocompact vector space* is a topological vector space that is complete and whose fundamental system of neighbourhoods of zero is formed by subspaces of finite codimension. Morphisms of pseudocompact vector spaces are continuous linear maps. A *graded pseudocompact vector space* is a graded object in the category of pseudocompact vector spaces, i.e. a sequence V^i, $i \in \mathbb{Z}$, where each V^i is a pseudocompact vector space with morphisms defined component-wise. Finally, a *dg pseudocompact vector space* is a graded pseudocompact vector space V^i, $i \in \mathbb{Z}$ with a continuous differential.

The category of pseudocompact vector spaces will be denoted by pcVect. The categories of dg pseudocompact vector spaces will be denoted by $\mathsf{pcDGVect}$.

PROPOSITION 2.2.2. *The category* Vect *is anti-equivalent to* pcVect*, and the category* $\mathsf{pcDGVect}$ *is anti-equivalent to* DGVect*.*

PROOF. Given a vector space V, its k-linear dual V^* is pseudocompact. Indeed, denoting by $\{V_\alpha\}$ the collection of finite-dimensional subspaces of V, we have $V = \varinjlim_\alpha V_\alpha$ and therefore $V^* = \varprojlim_\alpha V_\alpha^*$. So, V^* is complete with respect to the kernels of maps into finite-dimensional spaces. The functor backwards associates to a pseudocompact vector space V its *continuous* linear dual, also denoted by V^*. It is straightforward to see that this gives the desired anti-equivalence. The dg case is similar. □

The above proof shows that every (dg) pseudocompact vector space V is a projective limit of its finite dimensional (dg) quotients V_α: $V \cong \varprojlim_\alpha V_\alpha$. Conversely, a projective system of finite-dimensional dg vector spaces determines a dg pseudocompact vector space. Given two dg pseudocompact vector spaces $V \cong \varprojlim_\alpha V_\alpha$ and $U \cong \varprojlim_\beta U_\beta$ the dg (not pseudocompact in general) space of continuous maps

$V \to U$ is $\mathrm{Hom}(V,U) \cong \varprojlim_\beta \varinjlim_\alpha \mathrm{Hom}(V_\alpha, U_\beta)$. Indeed, finite-dimensional dg vector spaces are compact objects in DGVect, so by Proposition 2.2.2 they are the cocompact objects in $\mathsf{pcDGVect}$, i.e. the objects W such that $\mathrm{Hom}(-,W)$ sends projective limits to inductive colimits. Finally we note that a discrete vector space is pseudocompact if and only if it is finite-dimensional.

Recall that the category DGVect has a symmetric monoidal structure given by the usual tensor product. Similarly for two dg pseudocompact vector spaces $V = \varprojlim_\alpha V_\alpha$ and $U = \varprojlim_\beta U_\beta$ their completed tensor product is defined as $V \widehat{\otimes} U := \varprojlim_{\alpha,\beta} V_\alpha \otimes U_\beta$. We will omit the hat over the symbol of the tensor product as it will always be understood. With this definition the anti-equivalence of Proposition 2.2.2 is that of symmetric monoidal categories, i.e. there are natural isomorphisms $(V \otimes U)^* \cong V^* \otimes U^*$ where U and V are both dg vector spaces or both pseudocompact vector spaces.

2.2.2. Pseudocompact algebras, or coalgebras. Just as algebras and dg algebras can be defined succinctly as monoids in the symmetric monoidal categories Vect and DGVect, we now consider monoids in pcVect and $\mathsf{pcDGVect}$.

DEFINITION 2.2.3. A *pseudocompact* (*commutative*) *algebra* is a (commutative) monoid in the symmetric monoidal category pcVect. A *pseudocompact* (*commutative*) *differential graded algebra* is a (commutative) monoid in the symmetric monoidal category $\mathsf{pcDGVect}$.

These four categories are denoted by adding pc to their discrete versions; for example, the category of (graded) pseudocompact algebras is denoted by pcAlg. Each of them also admits an augmented version, which will be denoted by adding an asterisk at the end; for example, the category of augmented pseudocompact dg algebras is denoted by pcDGA^*.

One can also define a (cocommutative) coalgebra and a (cocommutative) dg coalgebra as a (cocommutative) comonoid in Vect and DGVect respectively. Using the monoidal anti-equivalence of Proposition 2.2.2, we see that the category of (cocommutative) coalgebras is anti-equivalent to category of pseudocompact (commutative) algebras. Similarly, the category of (cocommutative) dg coalgebras is anti-equivalent to category of pseudocompact (commutative) dg algebras.

The following result is a dg version of the so-called *fundamental theorem of coalgebras*.

THEOREM 2.2.4. *Any (cocommutative) dg coalgebra is a union of its finite-dimensional dg subcoalgebras.*

PROOF. The non-dg version of the theorem is well-known, see for example [Swe69, Theorem 2.2.1]. The dg version is an easy consequence since any (possibly non-differential) finite-dimensional subcoalgebra A of a dg coalgebra C is contained in the dg subcoalgebra $A \oplus d(A)$ which is also finite-dimensional. □

COROLLARY 2.2.5. *Any (commutative) pseudocompact dg algebra is the projective limit of its finite-dimensional quotients.*

PROOF. Given a (commutative) pseudocompact dg algebra A, its k-linear dual A^* is a (cocommutative) dg coalgebra. Then the desired statement is equivalent to saying that A^* is an inductive limit (i.e. a union) of its finite-dimensional dg subcoalgebras which is Theorem 2.2.4. □

REMARK 2.2.6. Theorem 2.2.4 (and hence, Corollary 2.2.5) uses the associativity condition in an essential way and does not hold for other algebraic structures, such as Lie coalgebras. See [Pos, Section 2.4] for an example of a Lie coalgebra possessing no proper Lie subcoalgebras at all.

We will consider *coaugmented* dg coalgebras, i.e. dg coalgebras C supplied with a dg coalgebra map $k \to C$. In this case the quotient C/k is a dg coalgebra without a counit. Given a dg coalgebra C we denote by $\Delta = \Delta^1 \colon C \to C \otimes C$ its comultiplication and by $\Delta^n \colon C \to C^{\otimes n}$ its nth iteration. A coaugmented dg coalgebra C is *conilpotent* if the reduced comultiplication $\bar{\Delta} \colon C/k \to C/k \otimes C/k$ in the non-counital dg coalgebra $(C/k, \bar{\Delta})$ satisfies $C/k = \bigcup_{n=1}^{\infty} \ker(\bar{\Delta}^n)$.

It is easy to see that C is conilpotent if and only if its dual pseudocompact dg algebra C^* is augmented and for its augmentation ideal I it holds that $C^* \cong \varprojlim_n C^*/I^n$. In other words, C^* is a local complete augmented dg algebra with the maximal dg ideal I. Note also that if an augmented pseudocompact dg algebra is local (i.e. its augmentation ideal I is a unique dg maximal ideal), then it is automatically I-adically complete since every ideal with finite-dimensional quotient must contain some power of I and so every finite-dimensional quotient factors through a power of I.

All together, we have the following result.

Proposition 2.2.7. *The category of (cocommutative) conilpotent dg coalgebras is anti-equivalent to the category of local augmented (commutative) pseudocompact dg algebras.* □

We will denote that latter category by $\mathsf{pcDGA_{loc}}$ and $\mathsf{pcCDGA_{loc}}$ in the commutative case.

Given a pseudocompact vector space V, the *completed tensor algebra on V* is

$$\widehat{T}(V) := \prod_{n=0}^{\infty} T^n(V), \text{ where } T^n(V) := V^{\widehat{\otimes} n}.$$

This is the free local pseudocompact algebra on V, in the sense that $V \mapsto \widehat{T}(V)$ defines a functor $\mathsf{pcVect} \to \mathsf{pcAlg_{loc}}$ that is left adjoint to the forgetful functor $\mathsf{pcAlg_{loc}} \to \mathsf{pcVect}$ sending a local pseudocompact algebra A to its underlying vector space:

$$\mathrm{Hom}_{\mathsf{pcAlg_{loc}}}(\widehat{T}(V), A) \cong \mathrm{Hom}_{\mathsf{pcVect}}(V, A).$$

There is also a 1-1 correspondence between derivations of $\widehat{T}(V)$ and (continuous) maps $V \to \widehat{T}V$.

Remark 2.2.8. This universal property does not hold when A is a non-local pseudocompact algebra, so $\widehat{T}V$ is not a free pseudocompact algebra. The construction of a free pseudocompact algebra is in general very different, and is discussed later in Section 5.1.

Similarly, the *completed symmetric algebra on V* is

$$\widehat{S}(V) := \prod_{n=0}^{\infty} S^n(V), \text{ where } S^n(V) := (V^{\widehat{\otimes} n})_{\mathbb{S}_n}.$$

This is the free local pseudocompact commutative algebra on V. Once again, we note that $\widehat{S}(V)$ is free only in $\mathsf{pcCAlg_{loc}}$. There is also a 1-1 correspondence between derivations of $\widehat{S}(V)$ and (continuous) maps $V \to \widehat{S}V$.

2.2.3. Pseudocompact modules, or comodules.

Definition 2.2.9. Given a pseudocompact dg algebra C, a (*right*) *pseudocompact C-module* is a pseudocompact vector space V together with a continuous linear map $V \otimes C \to V$ satisfying the usual axioms of associativity and unitality. The category of pseudocompact dg C-modules is denoted by $\mathsf{pcDGMod}\text{-}C$.

Just like the fundamental theorem of coalgebras, this category is anti-equivalent to the category of dg C^*-comodules, again via taking duals. Thus, all our results concerning pseudocompact dg modules can readily be translated into results about dg comodules if one wishes to do so.

2.3. Model categories

Model categories were introduced in [Qui67] (where they were referred to as "closed" model categories), as an abstraction of the category of topological spaces or simplicial sets. However it quickly became clear that this notion has much wider applicability, in particular, much of classical homological algebra can be formulated in the language of model categories. We will see in the next chapter that deformation theory can likewise be profitably recast in this language. The survey [DS95] covers most of our needs; for more in-depth treatment see [Hir03, Hov99].

2.3.1. Definition and examples of model categories. We begin by introducing some preliminary terminology for defining model categories.

Definition 2.3.1. Let C be a category and let i and p be maps in C such that there is a commutative solid diagram as below.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

A *lift* is a dashed map as in the diagram making the whole diagram commute. If a lift exists in the diagram for any f and g, then i is said to have the *left lifting property* (LLP) with respect to p, and p is said to have the *right lifting property* (RLP) with respect to i.

Definition 2.3.2. A *model category* is a category C with three distinguished classes of morphisms, called *weak equivalences* ($\mathcal{W}$, $\xrightarrow{\sim}$), *fibrations* ($\mathcal{F}$, $\twoheadrightarrow$) and *cofibrations* ($\mathcal{C}$, $\hookrightarrow$), each closed under compositions and containing all identity maps. A morphism is called an *acyclic* or *trivial* (*co*)*fibration* if it is both a (co)fibration and a weak equivalence. We require the axioms below to be satisfied.

- (MC1) *Completeness*: Limits and colimits exist in C.
- (MC2) *2-out-of-3*: If f and g are composable maps in C such that two of f, g and gf are weak equivalences, then so is the third.

(MC3) *Retracts*: The classes of morphisms $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$ are each closed under retracts.

(MC4) *Lifting*: Cofibrations have the LLP with respect to acyclic fibrations. Acyclic cofibrations have the LLP with respect to fibrations.

(MC5) *Factorization*: Any map f can functorially be factored in two ways:

(i) $f = pi$, where i is a cofibration and p is an acyclic fibration, and,

(ii) $f = pi$, where i is an acyclic cofibration and p is a fibration.

The above definition differs from the original one by Quillen in that the latter only assumes the existence of finite limits and colimits and the factorizations of maps as in axiom (MC5) were not required to be functorial. However, in practice, the strengthened axioms hold in most of the cases of interest and this modification is often preferred in the current literature.

Remark 2.3.3. The following observations on the axioms of a model category are simple but useful.

(1) The axioms are over-determined. That is, given a model category, the cofibrations are the morphisms that have the LLP with respect to acyclic fibrations, and the fibrations are the morphisms that have the RLP with respect to acyclic cofibrations.

(2) The axioms are self-dual. That is, given a model category C, its opposite category C^{op} is a model category where a morphism $f^{\mathsf{op}}\colon Y \to X$ is a fibration (resp. cofibration, weak equivalence) in C^{op} if its corresponding morphism $f\colon X \to Y$ is a cofibration (resp. fibration, weak equivalence) in C.

Due to existence of limits and colimits, a model category C has an initial object $\varnothing$ and a terminal object $*$; if these are isomorphic, C is called a *pointed* model category. An object X of C is *fibrant* if the unique map $X \to *$ is a fibration, and *cofibrant* if the unique map $\varnothing \to X$ is a cofibration. By the factorization axiom (MC5), every object X is functorially associated with a fibrant object RX and an acyclic cofibration $X \to RX$; similarly, X is functorially associated with a cofibrant object LX and an acyclic fibration $LX \to X$. We will call RX and LX *fibrant and cofibrant replacements* of X, respectively. Moreover, any object $X \in \mathsf{C}$ can be connected by (possibly a zigzag of) weak equivalences to an object that is both fibrant and cofibrant, for example, such is the object $L(RX)$ or $R(LX)$.

Example 2.3.4. Here are a few examples of model categories.

(1) The category Top of topological spaces is a model category where weak equivalences are the ordinary weak equivalences of topological spaces, fibrations are Serre fibrations and cofibrations are those maps that have the LLP with respect to Serre fibrations. All objects are fibrant and the cofibrant objects are retracts of CW complexes. This is the prototypical model category that served as a blueprint and motivation for developing the whole theory of model categories.

(2) The category $\mathsf{Ch}(R)$ of chain complexes of modules over an associative ring R has two natural model category structures with weak equivalences being quasi-isomorphisms of chain complexes. In the first one (called the *projective model structure*) fibrations are surjective maps and cofibrations are chain maps having the LLP with respect to surjective chain maps, whereas in the second one (called the *injective model structure*) cofibrations are injective maps and fibrations are chain maps having the RLP with respect to injective chain maps.

(3) The categories CDGA and DGLA of commutative dg algebras and dg Lie algebras over a field of characteristic zero and DGA and DGA^* of dg algebras and *augmented* dg algebras over a field of arbitrary characteristic have model structures, given in [Hin97]. Weak equivalences are quasi-isomorphisms, fibrations are surjective maps and cofibrations are the maps having the LLP with respect to fibrations. All objects are fibrant in these model categories.

2.3.2. Homotopy categories. Next we discuss the notion of homotopy. In model categories, homotopies come in two flavours, namely "left" and "right" homotopies, which are based on "cylinder objects" and "path objects" respectively. The idea is that for sufficiently nice objects, such as CW complexes in topological spaces, the two notions of homotopy coincide.

DEFINITION 2.3.5. Let C be a model category and $X \in \mathsf{C}$. A *cylinder object* for X is an object $X \times I$ in C with a factorization of the canonical fold map $\nabla_X = \mathrm{id}_X + \mathrm{id}_X$ into

$$X \amalg X \xrightarrow{i} X \times I \xrightarrow[\sim]{p} X, \qquad p \circ i = \nabla_X,$$

with i a cofibration and p an acyclic fibration. Let $f, g\colon X \to Y$ be two maps in C. A *left homotopy* from f to g is a map $H\colon X \times I \to Y$ for some cylinder object $X \times I$ for X making the following diagram commute:

$$\begin{array}{ccc} X \amalg X & \xrightarrow{f+g} & Y \\ {\scriptstyle i}\downarrow & \nearrow_{H} & \\ X \times I & & \end{array}$$

If such a left homotopy exists, then f and g are *left homotopic*, denoted by $f \simeq_l g$.

DEFINITION 2.3.6. Let C be a model category and $Y \in \mathsf{C}$. A *path object* for Y is an object Y^I in C with a factorization of the canonical diagonal map $\Delta_Y = (\mathrm{id}_Y, \mathrm{id}_Y)$ into

$$Y \xrightarrow[\sim]{i} Y^I \xrightarrow{p} Y \times Y, \qquad p \circ i = \Delta_Y,$$

with i an acyclic cofibration and p a fibration. Let $f, g\colon X \to Y$ be two maps in C. A *right homotopy* from f to g is a map $H\colon X \to Y^I$ for some path object Y^I for Y making the following diagram commute:

$$\begin{array}{ccc} & & Y^I \\ & {}^{H}\nearrow & \downarrow{\scriptstyle p} \\ X & \xrightarrow[(f,g)]{} & X \times X \end{array}$$

If such a right homotopy exists, then f and g are *right homotopic*, denoted by $f \simeq_r g$.

REMARK 2.3.7. Some authors prefer to weaken the notions of a cylinder and path object, for example, not insisting that the map $X \amalg X \to X \times I$ be a cofibration (note that in the case of topological spaces the standard topological cylinder $X \times [0, 1]$ this condition is not satisfied unless X is a CW complex). Nevertheless, the factorization axiom (MC5) ensures that any object has a functorial cylinder and path object.

The following result holds.

THEOREM 2.3.8. *Let X be a cofibrant object and Y be a fibrant object of a model category C. Then*

(1) *Two maps $X \to Y$ are left homotopic if and only if they are right homotopic.*
(2) *The relation of left or right homotopy on $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ is an equivalence relation. The set of homotopy classes of maps $X \to Y$ will be denoted by $[X, Y]$.*
(3) *If $f, g\colon X \to Y$ are left homotopic and $h\colon A \to X$ is a map with A cofibrant, then $h \circ f$ and $h \circ g$ are left homotopic. Similarly if $k\colon Y \to B$ is a map with B fibrant then $f \circ k$ and $g \circ k$ are right homotopic.*
(4) *If X' is a cofibrant object weakly equivalent to X and Y' is a fibrant object weakly equivalent to Y then there is a bijection $[X, Y] \cong [X', Y']$.*
(5) *Suppose additionally that $X, Y \in \mathsf{C}$ are both fibrant and cofibrant and that $f\colon X \to Y$ is a weak equivalence. Then X and Y are* homotopy equivalent*, i.e. there exists a map $g\colon Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X.*

PROOF. See [Hov99, Proposition 1.2.5] for parts (1)–(4), and [Hov99, Proposition 1.2.8] for part (5). □

This allows one to construct the homotopy category of a model category.

DEFINITION 2.3.9. The *homotopy category* of a model category C is the category $\mathsf{Ho}\,\mathsf{C}$ whose objects are the objects in C that are both fibrant and cofibrant and for two fibrant-cofibrant objects $X, Y \in \mathsf{C}$ we have $\mathrm{Hom}_{\mathsf{Ho}\,\mathsf{C}}(X, Y) := [X, Y]$, the homotopy classes of maps from X to Y.

Theorem 2.3.8 ensures that $\mathsf{Ho}\,\mathsf{C}$ is well-defined. Moreover, the correspondence $X \mapsto L(RX)$ (or, equivalently, $X \mapsto R(LX)$) determines a functor $\gamma\colon \mathsf{C} \to \mathsf{Ho}\,\mathsf{C}$. It follows from Theorem 2.3.8(5) that γ takes weak equivalences in C into isomorphisms in $\mathsf{Ho}\,\mathsf{C}$; remarkably, γ is the *universal functor* out of C having this property.

THEOREM 2.3.10. *Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor from a model category C to a category D such that for any weak equivalence $f \in \mathsf{C}$ its image $F(f) \in \mathsf{D}$ is an isomorphism. Then there exists a unique functor $G\colon \mathsf{Ho}\,\mathsf{C} \to \mathsf{D}$ such that $G \circ \gamma = F$.*

PROOF. See [DS95, Theorem 6.2]. □

Remark 2.3.11. The homotopy category of a model category C is where the most important invariants of C lie. For example, the *derived category* of a ring R is the homotopy category of $\mathsf{Ch}(R)$ from Example 2.3.4(2), with either projective or injective model structure. Thus, different model structures on the same category may lead to equivalent homotopy categories.

2.3.3. Quillen functors. Having defined the notion of a model category, it is natural to consider functors between different model categories. It is unreasonable to require that functors preserve the whole structure available (i.e. all classes $\mathcal{W}$, $\mathcal{F}$, $\mathcal{C}$) as this does not hold in many cases of interest. The appropriate notion here is that of a "Quillen adjunction".

Definition 2.3.12. Let C and D be model categories. An adjunction

$$F\colon \mathsf{C} \rightleftarrows \mathsf{D} : G,$$

with F left adjoint to G, is a *Quillen adjunction* if F preserves cofibrations and G preserves fibrations. In this case, F is called a *left Quillen functor* and G is called a *right Quillen functor*.

If $F\colon \mathsf{C} \rightleftarrows \mathsf{D} : G$ is a Quillen adjunction, then one can prove that F carries weak equivalences between cofibrant objects into weak equivalences and likewise G carries weak equivalences between fibrant objects into weak equivalences. It follows that F and G lift to functors LF and RG between the corresponding homotopy categories $\mathsf{Ho}\,\mathsf{C}$ and $\mathsf{Ho}\,\mathsf{D}$. We will refer to LF as the *left derived functor* of F and to RG as the *right derived functor* of G. Moreover, (LF, RG) also form an adjoint pair:

Theorem 2.3.13. *Any Quillen adjunction* $F\colon \mathsf{C} \rightleftarrows \mathsf{D} : G$ *induces an (ordinary) adjunction*

$$LF\colon \mathsf{Ho}\,\mathsf{C} \rightleftarrows \mathsf{Ho}\,\mathsf{D} : RG.$$

Proof. See [DS95, Theorem 9.7]. □

Definition 2.3.14. A Quillen adjunction $F\colon \mathsf{C} \rightleftarrows \mathsf{D} : G$ is called a *Quillen equivalence* if the corresponding adjunction $LF\colon \mathsf{Ho}\,\mathsf{C} \rightleftarrows \mathsf{Ho}\,\mathsf{D} : RG$ is an ordinary equivalence.

EXAMPLE 2.3.15.

(1) Let R be an associative ring and C be the category of chain complexes of R modules with its projective model structure, and D be the same category with the injective model structure, cf. Example 2.3.4(2). Then the identity functor $\mathsf{C} \to \mathsf{D}$ is a right Quillen functor establishing a Quillen equivalence between C and D. Its adjoint left Quillen functor $\mathsf{D} \to \mathsf{C}$ is, of course, also the identity functor. Informally, this can be interpreted as saying that there are two equivalent approaches to classical homological functors: one based on injective resolutions and the other based on projective resolutions.

(2) The functor of geometric realization from simplicial sets to topological spaces is a left Quillen functor whose right adjoint is the functor associating to a topological space its singular simplicial set [Hov99]. This adjunction is a Quillen equivalence.

(3) In the next chapter, we will consider Koszul duality as a Quillen equivalence between the categories of commutative dg algebras with the model structure of Example 2.3.4(3) and the category of pseudocompact dg Lie algebras and see that it underlies the modern approach to deformation theory.

The following criterion is useful for showing that a Quillen adjunction is a Quillen equivalence.

THEOREM 2.3.16. *Let* $F\colon \mathsf{C} \rightleftarrows \mathsf{D} : G$ *be a Quillen adjunction. The following are equivalent:*

(1) (F, G) *is a Quillen equivalence.*

(2) F *reflects weak equivalences* (*in the sense that, if* Ff *is a weak equivalence in* D*, then* f *is a weak equivalence in* C) *between cofibrant objects, and, for all fibrant* $Y \in \mathsf{D}$*, the composition* $FLGY \to FGY \to Y$ *is a weak equivalence in* D*.*

(3) G *reflects weak equivalences between fibrant objects, and, for all cofibrant* $X \in \mathsf{C}$*, the composition* $X \to GFX \to GRFX$ *is a weak equivalence in* C*.*

PROOF. See [Hov99, Corollary 1.3.16]. □

2.3.4. Cofibrantly generated model categories. We collect some definitions and results on cofibrantly generated model categories from [Hov99].

Definition 2.3.17. Let C be a category with and I be a class of maps in C.

(1) A *λ-sequence in I* is a functor $X\colon \lambda \to \mathsf{C}$, i.e. a diagram
$$X_0 \to X_1 \to \cdots \to X_\beta \to \dots,$$
such that each map $X_\beta \to X_{\beta+1}$ ($\beta < \lambda$) is in I and such that X is colimit-preserving, i.e. the induced map $\operatorname{colim}_{\beta<gamma} X_\beta \to X_\gamma$ is an isomorphism for any limit ordinal $\gamma < \lambda$.

(2) A *transfinite composition* of a λ-sequence X in I is the map
$$X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta.$$

(3) An object $A \in \mathsf{C}$ is *small relative to I* if there is some cardinal κ such that for any κ-filtered ordinal λ and any λ-sequence X in I, the induced map
$$\operatorname{colim}_{\beta<\lambda} \operatorname{Hom}_{\mathsf{C}}(A, X_\beta) \to \operatorname{Hom}_{\mathsf{C}}(A, \operatorname{colim}_{\beta<\lambda} X_\beta).$$
is a set bijection. The object A is *small* if it is small relative to C.

Definition 2.3.18. Let C be a category with all small colimits and limits and I be a class of maps in C.

(1) A morphism is called *I-injective* (resp. *I-projective*) if it has the right (resp. left) lifting property with respect to morphisms in I. We write
$$I\text{-inj} := \mathrm{RLP}(I) \text{ and } I\text{-proj} := \mathrm{LLP}(I).$$

(2) A morphism is called an *I-fibration* (resp. *I-cofibration*) if it has the right (resp. left) lifting property with respect to I-projective (resp. I-injective) morphisms. We write
$$I\text{-fib} := \mathrm{RLP}(I\text{-proj}) \text{ and } I\text{-cof} := \mathrm{LLP}(I\text{-inj}).$$

(3) A map is a *relative I-cell complex* if it is a transfinite composition of pushouts of elements of I. We denote by I-cell the class of relative I-cell complexes.

Definition 2.3.19. A model category C is said to be *cofibrantly generated* if there are sets I and J of maps such that the following conditions hold.

(1) The domains of the maps of I are small relative to I-cell.
(2) The domains of the maps of J are small relative to J-cell.
(3) Fibrations are J-injective.

(4) Trivial fibrations are I-injective.

The set I is called the *set of generating cofibrations*, and J the *set of generating trivial cofibrations*.

2.3.5. Proper model categories. In a model category C one can define the notions of homotopy pullbacks and homotopy pushouts; an elementary construction can be found in [DS95, Section 10].

Definition 2.3.20. Let X, Y and Z be objects in a model category C supplied with maps $X \to Y$ and $X \to Z$. Factor the map $LX \to X \to Y$ as $LX \xrightarrow{i_1} \tilde{Y} \xrightarrow{p_1} Y$ where i_1 is a cofibration and p_1 is an acyclic fibration; similarly factor the map $LX \to X \to Z$ as $LX \xrightarrow{i_2} \tilde{Z} \xrightarrow{p_2} Z$ where i_2 is a cofibration and p_2 is an acyclic fibration. Then the *homotopy pushout* $Y \amalg_X^h Z$ is by definition $\tilde{Y} \amalg_{LX} \tilde{Z}$.

A *homotopy pullback* is defined dually as a homotopy pushout in C^{op}. It will be denoted for objects X,Y and Z by $Y \times_X^h Z$.

Remark 2.3.21. The notions of a homotopy pullbacks and pushout are derived functors of ordinary pullbacks and pushouts. Namely, consider the category of diagrams $\mathsf{Push}(\mathsf{C})$ in a model category C of the form $Y \leftarrow X \to Z$ and a functor $F\colon \mathsf{Push}(\mathsf{C}) \to \mathsf{C}$ obtained by taking the pushout of a given diagram. Then there exists a model structure on $\mathsf{Push}(\mathsf{C})$ such that F is a left Quillen functor and then the homotopy pushout is its left derived functor. The case of a homotopy pullback is similar.

Homotopy pushouts and pullbacks are simplified in *proper* model categories.

Definition 2.3.22. A model category C is called *left proper* if for any pushout diagram in C

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \longrightarrow & D \end{array}$$

for which i is a cofibration and f is a weak equivalence, then the map g is also a weak equivalence. Dually, C is *right proper* if for any pullback diagram in C

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[p]{} & D \end{array}$$

for which p is a fibration and g is a weak equivalence, then the map f is also a weak equivalence.

Many model categories are left or right proper, as the following result makes clear.

PROPOSITION 2.3.23. *Let C be a model category such that every object of C is cofibrant. Then C is left proper. Dually, if every object of C is fibrant, then C is right proper.*

PROOF. See [Lur, Proposition A.2.4.2]. □

Then the following result holds.

PROPOSITION 2.3.24. *Let $Y \leftarrow X \rightarrow Z$ be a diagram in a left proper model category where $X \rightarrow Y$ is a cofibration. Then $Y \amalg_X Z$ is weakly equivalent to $Y \amalg_X^h Z$.*

Dually, let $Y \rightarrow X \leftarrow Z$ be a diagram in a right proper model category where $Z \rightarrow X$ is a fibration. Then $Y \times_X Z$ is weakly equivalent to $Y \times_X^h Z$.

PROOF. See [Lur, Proposition A.2.4.4]. □

Lastly, we discuss the existence of derived mapping spaces in model categories.

THEOREM 2.3.25. *Let X be a cofibrant object and Y be a fibrant object in a model category C.*

(1) *For any object A, there exists a simplicial set $\mathrm{Map}_l(A, Y)$ such that*

$$\pi_0 \mathrm{Map}_l(A, Y) \cong [A, Y]_l,$$

and a simplicial set $\mathrm{Map}_r(X, A)$ such that

$$\pi_0 \mathrm{Map}_r(X, A) \cong [X, A]_r.$$

(2) *The functors $A \mapsto \mathrm{Map}_l(A, Y)$ and $A \mapsto \mathrm{Map}_r(X, A)$ are left and right Quillen functors from C to simplicial sets respectively.*

(3) *There is a natural isomorphism $\mathrm{Map}_l(X, Y) \cong \mathrm{Map}_r(X, Y)$.*

PROOF. See [Hov99, Section 5.4]. □

When X is cofibrant and Y is fibrant, we will write $\mathrm{Map}(X, Y)$ for either $\mathrm{Map}_l(X, Y)$ or $\mathrm{Map}_r(X, Y)$ and call it the *derived mapping space* from X to Y.

CHAPTER 3

Homotopical approach to deformation theory

In this chapter, we prove a version of Deligne's principle: in characteristic zero, every deformation problem is governed by a dg Lie algebra. This is a central result in deformation theory and there have been many formalizations of this result, for example, [Pri10] which uses simplicial techniques, and [Lur] where this principle is formulated in the framework of ∞-categories. Our proof has the advantage of being more elementary than those in the aforementioned references.

More precisely, we show that any extended deformation functor (a set-valued functor satisfying certain conditions) from the category of local pseudocompact commutative dg algebras is isomorphic to a functor of the form

$$\mathscr{MC}(L,-) := \mathrm{MC}(L,-)/\sim$$

for some dg Lie algebra L. Here $\mathrm{MC}(L) = \{x \in L^1 : dx + \frac{1}{2}[x,x] = 0\}$ denotes the set of *Maurer–Cartan elements* of a dg Lie algebra L, and $\sim$ denotes *gauge equivalence*. We also prove a noncommutative version of this result that is valid in any characteristic.

3.1. Brown representability for compactly generated model categories

The Brown representability theorem [Bro62] is a necessary and sufficient condition for a functor defined on the homotopy category of pointed topological spaces to be representable. It has subsequently been formulated in various abstract contexts. It will be convenient for us to use a version formulated in [Jar11] for compactly generated model categories. Recall that an object K in a category C is said to be *compact* if the functor $\mathrm{Hom}_{\mathsf{C}}(K,-)\colon \mathsf{C} \to \mathsf{Set}$ preserves inductive colimits.

Definition 3.1.1. Let C be a model category. We say that C is *compactly generated* if there exists a set S of compact cofibrant objects in C that *detect weak equivalences*, that is, a map $f\colon X \to Y$ in C is a weak equivalence if and only if f induces a bijection $[K,X] \to [K,Y]$ for any $K \in S$. The elements of S are called *compact generators* for C.

Example 3.1.2. The category of connected pointed topological spaces is compactly generated with $S := \{S^n : n = 1, 2, \dots\}$, the pointed spheres. It is interesting to note that the category of all (i.e. not necessarily connected) topological spaces is not compactly generated, [Hel81].

Remark 3.1.3. There is another, inequivalent notion of a compactly generated model category contained in e.g. [MP12]. Under this notion the category of all topological spaces *is* compactly generated.

Under the assumption of compact generation, an abstract Brown representability holds in C.

Theorem 3.1.4. *Let C be a compactly generated pointed model category with $*$ denoting its initial-terminal object. Suppose that a set-valued contravariant functor F on C satisfies the following conditions:*

1. *$F(*) = *$.*
2. *F takes weak equivalences to bijections of sets.*
3. *F takes arbitrary coproducts of cofibrant objects in C into products of sets.*
4. *Let A, B, C be cofibrant objects in C and $A \to B$, $A \to C$ be morphisms in C with $A \to B$ being a cofibration. Then the natural map*

$$F(B \textstyle\coprod_A C) \to F(B) \times_{F(A)} F(C)$$

is a surjection of sets.

Then the functor F is representable in the homotopy category of C, i.e. there exists an object X in C and a natural weak equivalence $F(Y) \simeq [Y, X]$ for any $Y \in \mathsf{C}$.

Proof. This is [Jar11, Theorem 19]. □

Remark 3.1.5. Theorem 3.1.4 is a model category version of the famous Brown representability theorem [Bro62] that was originally formulated in the category of pointed CW complexes. It is not the most general form of Brown's representability theorem (for such a statement see [Hel81]) since it can be formulated in a way not requiring the existence of a model structure. In practice (and particularly for the application we have in mind) a model structure is often present and the conditions of the theorem are usually not difficult to verify.

Remark 3.1.6. It is easy to see that, conversely, a representable up to homotopy set-valued functor on a compactly generated model category must satisfy the conditions listed in Theorem 3.1.4. To make a comparison with topology easier, we will view F as a *covariant* functor on C^{op} represented by $X \in \mathsf{C}^{\mathsf{op}}$; we will assume without loss of generality that X is cofibrant. Thus, for $Y \in \mathsf{C}^{\mathsf{op}}$ we have $F(Y) = [X, Y]$. The conditions (1), (2) and (3) are obvious. Applying $\operatorname{Map}(X, -)$ to a homotopy pullback of $B \to A \leftarrow C$ in C, we obtain a homotopy pullback of simplicial sets (since $\operatorname{Map}(X, -)$ is a right Quillen functor).

$$\begin{array}{ccc} \operatorname{Map}(X, B \times^h_A C) & \longrightarrow & \operatorname{Map}(X, B) \\ \downarrow & & \downarrow \\ \operatorname{Map}(X, C) & \longrightarrow & \operatorname{Map}(X, A) \end{array}$$

Taking the connected components functor, we obtain a surjection

$$\begin{array}{ccc} \pi_0 \operatorname{Map}(X, B \times^h_A C) & \longrightarrow & \pi_0 \operatorname{Map}(X, B) \times_{\pi_0 \operatorname{Map}(X,A)} \pi_0 \operatorname{Map}(X, B) \\ \wr\| & & \wr\| \\ F(B \times^h_A C) & & F(B) \times_{F(A)} F(C) \end{array}$$

as required.

Note also that this argument shows that one should not, in general, expect that the map $F(B \times^h_A C) \to F(B) \times_{F(A)} F(C)$ is an isomorphism. Indeed, it follows from the homotopy pullback diagram above that the homotopy fibre of the map

$$\operatorname{Map}(X, B \times^h_A C) \to \operatorname{Map}(X, B) \times \operatorname{Map}(X, C)$$

over a given point $(f, g) \in \operatorname{Map}(X, B) \times \operatorname{Map}(X, C)$ having the same image in $[X, A]$ is the based loop space $\Omega \operatorname{Map}(X, A)$. Thus, the fibration

$$\Omega \operatorname{Map}(X, A) \to \operatorname{Map}(X, B \times^h_A C) \to \operatorname{Map}(X, B) \times \operatorname{Map}(X, C)$$

gives rise to a long homotopy exact sequence (the Mayer-Vietoris sequence, [DR80])

$$\begin{array}{ccccc} \cdots \longrightarrow \pi_1 \operatorname{Map}(X, B) \times \pi_1 \operatorname{Map}(X, C) & \longrightarrow & \pi_1 \operatorname{Map}(X, A) & & \\ & & \downarrow & & \\ & & F(B \times^h_A C) & \longrightarrow & F(B) \times_{F(A)} F(C). \end{array}$$

3.2. Maurer–Cartan elements and Maurer–Cartan moduli sets

We will outline here the general theory of Maurer–Cartan elements in dg Lie and associative algebras and related moduli sets. We defer the more general definition of Maurer–Cartan elements in L_∞- and A_∞-algebras to the next chapter, as one requires completeness conditions to ensure that certain infinite series converge. Such technicalities are not required in this chapter.

3.2.1. Maurer–Cartan moduli in dg Lie algebras.

DEFINITION 3.2.1. Let $\mathfrak{g}$ be a dg Lie algebra over a field k of characteristic zero. An element $x \in \mathfrak{g}^1$ is called an *Maurer–Cartan element* if it satisfies the following equation (called the Maurer–Cartan or master equation)

$$d(x) + \frac{1}{2}[x, x] = 0.$$

The set of Maurer–Cartan elements in $\mathfrak{g}$ will be denoted by $\mathrm{MC}(\mathfrak{g})$. If A is a commutative dg algebra then $\mathfrak{g} \otimes A$ has naturally the structure of a dg Lie algebra and we will write $\mathrm{MC}(\mathfrak{g}, A)$ for $\mathrm{MC}(\mathfrak{g} \otimes A)$.

From now on we shall assume that $\mathfrak{g}$ is nilpotent or, more generally, pro-nilpotent (i.e. $\mathfrak{g} \cong \varprojlim_n \mathfrak{g}/\mathfrak{g}^{[n]}$ where $\mathfrak{g}^{[n]}$ is the dg Lie ideal generated by Lie products of at least n elements). In this case it has a group G associated to it. To define G, recall that $U\mathfrak{g}$, the universal enveloping algebra of $\mathfrak{g}$, is the graded associative algebra obtained by quotienting out the tensor algebra $T\mathfrak{g}$ by the ideal generated by the relations $a \otimes b - (-1)^{|a||b|} b \otimes a - [a, b]$ for two homogeneous elements $a, b \in \mathfrak{g}$. By definition there is a map $\mathfrak{g} \to U\mathfrak{g}$ that turns out to be an embedding. The algebra $U\mathfrak{g}$ is a bialgebra with the elements of $\mathfrak{g} \subset U\mathfrak{g}$ being primitive elements; moreover the set of primitive elements in $U\mathfrak{g}$ coincides with $\mathfrak{g}$. There is also an augmentation $U\mathfrak{g} \to k$ that sends all elements of $\mathfrak{g}$ to zero.

We will need to consider the completion $\widehat{U}\mathfrak{g}$ of $U\mathfrak{g}$ at its augmentation ideal $\mathfrak{I}$; i.e. $\widehat{U}\mathfrak{g} \cong \varprojlim_n U\mathfrak{g}/\mathfrak{I}^n$. Note that for a general dg Lie algebra $\mathfrak{g}$ it may happen that $\widehat{U}\mathfrak{g} = 0$, such is the case, e.g. when $\mathfrak{g}$ is an ordinary semisimple Lie algebra. However when $\mathfrak{g}$ is pro-nilpotent, $\widehat{U}\mathfrak{g}$ is always nontrivial; moreover the natural map $U\mathfrak{g} \to \widehat{U}\mathfrak{g}$ is an embedding and so, $\mathfrak{g}$ is likewise a subspace of $\widehat{U}\mathfrak{g}$. Then we define the group G as the group of group-like elements in the bialgebra $\widehat{U}\mathfrak{g}$, i.e. the set of elements $g \in \widehat{U}\mathfrak{g}$ such that $\Delta(g) = g \otimes g$ for the comultiplication Δ on $\widehat{U}\mathfrak{g}$.

There is, in fact, an equivalence of categories between pro-nilpotent Lie algebras, pro-nilpotent Lie groups and complete cocommutative Hopf algebras, cf. [Qui67, Appendix A3].

The group G is called the *gauge group* and acts on $\mathrm{MC}(\mathfrak{g})$ by gauge transformations:

PROPOSITION 3.2.2. *Let $g \in G$ and $x \in \mathrm{MC}(\mathfrak{g})$. Both elements g and x are viewed as lying in $\widehat{U}\mathfrak{g}$. Then the formula $g \cdot x := gxg^{-1} - d(g)g^{-1}$ determines an action of G on $\mathrm{MC}(\mathfrak{g})$.*

PROOF. First note that if $\mathfrak{g}$ has vanishing differential then the Maurer–Cartan condition takes the form $[x, x] = 0$ and the gauge action reduces to ordinary conjugation; the desired statement in this case is clear. We will reduce the general case to this one as follows. Introduce the graded Lie algebra $\tilde{\mathfrak{g}}$ having underlying graded vector space $\mathfrak{g} \oplus k \cdot d$ where $k \cdot d$ is the one-dimensional Lie algebra spanned by a symbol d sitting in cohomological degree 1. By definition for $a \in \tilde{\mathfrak{g}}$ we have $[d, a] := d(a)$, $[d, d] = 0$ whereas $\mathfrak{g}$ is a Lie subalgebra in $\tilde{\mathfrak{g}}$. Given $y \in \mathfrak{g}$ denote by $\tilde{y}$ the element $y + d \in \tilde{\mathfrak{g}}$. A straightforward check shows that a degree 1 element $x \in \mathfrak{g}$ is Maurer–Cartan if and only if $[\tilde{x}, \tilde{x}] = 0$. We will view an element $g \in \mathfrak{g}$ as an element in $\widehat{U}\tilde{\mathfrak{g}}$ via the embedding $\mathfrak{g} \subset \tilde{\mathfrak{g}} \subset \widehat{U}\tilde{\mathfrak{g}}$. Since $d(g) = [d, g] = dg - gd \in \widehat{U}\tilde{g}$ we have $d(g)g^{-1} = d - gdg^{-1}$ and so

$$
\begin{aligned}
g\tilde{x}g^{-1} &= g(x + d)g^{-1} \\
&= gxg^{-1} + gdg^{-1} \\
&= gxg^{-1} + d - d(g)g^{-1} \\
&= \widetilde{g \cdot x}.
\end{aligned}
$$

So, any Maurer–Cartan element $x \in \mathfrak{g}$ gives rise to an Maurer–Cartan element $\tilde{x} \in \tilde{\mathfrak{g}}$ where $\tilde{\mathfrak{g}}$ has vanishing differential and the gauge action in $\mathfrak{g}$ corresponds to the conjugation action in $\tilde{\mathfrak{g}}$. The desired statement is now obvious. □

Two Maurer–Cartan elements $x, y \in \mathfrak{g}$ are said to be *gauge equivalent* if $x = g \cdot y$ for some $g \in G$. We use $\sim$ to denote the corresponding equivalence relation.

DEFINITION 3.2.3. Given a pro-nilpotent dg Lie algebra $\mathfrak{g}$ we define its *Maurer–Cartan moduli set* $\mathscr{MC}(\mathfrak{g})$ as the set of equivalence classes $\mathrm{MC}(\mathfrak{g})/\sim$ under gauge equivalence.

If A is a commutative dg algebra, we will write $\mathscr{MC}(\mathfrak{g}, A)$ for $\mathscr{MC}(\mathfrak{g} \otimes A)$.

Let us now discuss the important notion of *homotopy* of Maurer–Cartan elements. First, let $k[t, dt]$ be the graded commutative k-algebra generated by one polynomial generator t in degree 0 and one exterior generator dt in degree 1. The differential is defined by the rule $d(t) = dt$ and extended to the whole $k[t, dt]$ by the Leibniz rule. Note that there are two maps $k[t, dt] \to k$ given by setting $t = 0$ or $t = 1$. Note that $k[t, dt]$ is a path object for k in the model category CDGA of commutative dg algebras. Note also that for any dg Lie algebra $\mathfrak{g}$ the tensor product $\mathfrak{g} \otimes k[t, dt] =: \mathfrak{g}[t, dt]$ is a dg Lie algebra and evaluations at 0 and 1 determine two dg Lie algebra maps $\mathfrak{g}[t, dt] \to \mathfrak{g}$.

Definition 3.2.4. Let $\mathfrak{g}$ be a nilpotent dg Lie algebra. Two Maurer–Cartan elements $x, y \in \mathfrak{g}$ are called *Sullivan homotopic* if there exists $z \in \mathrm{MC}(\mathfrak{g}[t, dt])$ such that $z|_{t=0} = x$ and $z|_{t=1} = y$.

An important theorem due to Schlessinger and Stasheff [SS] shows that homotopy and gauge equivalence are equivalent notions for nilpotent dg Lie algebras.

Theorem 3.2.5. *Let $\mathfrak{g}$ be a nilpotent dg Lie algebra. Then two Maurer–Cartan elements $x, y \in \mathfrak{g}$ are Sullivan homotopic if and only if they are gauge equivalent. In particular, the relation of homotopy on $\mathrm{MC}(\mathfrak{g})$ is an equivalence relation.*

Proof. See, e.g. [CL10, Theorem 4.4]. A generalization of the theorem was also independently proved in [Vor12, Theorem 5.2], in the context of arbitrary odd elements in dg Lie superalgebras with an associated gauge group. ☐

Remark 3.2.6. The construction $\mathfrak{g}[t, dt] := \mathfrak{g} \otimes k[t, dt]$ used in the definition of Sullivan homotopy makes sense for any dg Lie algebra $\mathfrak{g}$. For a general dg Lie algebra $\mathfrak{g}$ one does not expect to get a reasonable definition of an equivalence of Maurer–Cartan elements in $\mathfrak{g}$ using this construction. Suppose that $\mathfrak{g}$ is pro-nilpotent, in that case we define $\mathfrak{g}[t, dt] := \varprojlim_n (\mathfrak{g}/\mathfrak{g}^{[n]}[t, dt])$ and modify the notion of homotopy of Maurer–Cartan elements accordingly. It is easy to see that Schlessinger–Stasheff theorem 3.2.5 remains valid in this context. Moreover, Theorem 3.2.5 has a natural interpretation in terms of model categories: it says, roughly speaking, that the notions of left and right homotopy for nilpotent dg Lie algebras agree (see Chapter 4 for a precise statement and its generalizations).

3.2.2. Maurer–Cartan moduli in dg algebras. We will now outline a parallel treatment of Maurer–Cartan moduli for associative augmented dg algebras. It will be convenient for us to work with *non-unital* dg algebras, i.e. dg algebras not necessarily possessing a unit. It is well-known that the categories of non-unital dg algebras and of *augmented* dg algebras are equivalent: given a non-unital dg-algebra $\mathfrak{g}$ one can adjoin a unit forming an augmented dg algebra $\mathfrak{g}_e := \mathfrak{g} \oplus k \cdot 1$, and conversely, any augmented dg algebra gives rise to a non-unital dg algebra, its augmentation ideal.

Definition 3.2.7. Let $\mathfrak{g}$ be a non-unital dg algebra over a field k of arbitrary characteristic. An element $x \in \mathfrak{g}$ is called an *Maurer–Cartan element* if it satisfies $d(x) + x^2 = 0$. The set of all Maurer–Cartan elements in $\mathfrak{g}$ will be denoted by $\mathrm{MC}(\mathfrak{g})$.

Assume from now on that the non-unital dg algebra $\mathfrak{g}$ is *pro-nilpotent*. In other words, we have $\mathfrak{g} = \varprojlim_n \mathfrak{g}/\mathfrak{g}^{[n]}$; here $\mathfrak{g}^{[n]}$ is the dg ideal of $\mathfrak{g}$ generated by products of at least n elements. Clearly the elements of $\mathfrak{g}_e$ of the form $1 + i$ where $i \in \mathfrak{g}^0$, are invertible, and therefore form a group G that we will call the *gauge group* associated to $\mathfrak{g}$.

Proposition 3.2.8. *Let $g \in G$ and $x \in \mathrm{MC}(\mathfrak{g})$. Then the formula $g \cdot x := gxg^{-1} - d(g)g^{-1}$ determines an action of G on $\mathrm{MC}(\mathfrak{g})$.*

This action is well-defined by a similar argument as for Proposition 3.2.2; this time we should make use of the associative algebra $\tilde{\mathfrak{g}}$ having underlying space $\mathfrak{g} \otimes k[d]$ where d is a degree one element with $d^2 = 0$ (so that $k[d]$ is the exterior algebra on d which can be viewed as the universal enveloping algebra of the abelian Lie algebra $k \cdot d$). The product in $\tilde{\mathfrak{g}}$ is determined by requiring that $\mathfrak{g}$ and $k[d]$ are subalgebras in $\tilde{\mathfrak{g}}$ and there is a commutation relation $[d, a] = da - (-1)^{|a|}ad = d(a)$ for a being a homogeneous element in $\mathfrak{g}$ of degree $|a|$. As before, we say that two Maurer–Cartan elements $x, y \in \mathfrak{g}$ are *gauge equivalent* if $x = g \cdot y$ for some $g \in G$ and let $\sim$ denote the corresponding equivalence relation.

Definition 3.2.9. The *Maurer–Cartan moduli set* $\mathscr{MC}(\mathfrak{g})$ is the set of equivalence classes $\mathrm{MC}(\mathfrak{g})/\sim$ under gauge equivalence.

As before, if A is another dg algebra then we write $\mathrm{MC}(\mathfrak{g}, A)$ for $\mathrm{MC}(\mathfrak{g} \otimes A)$, and write $\mathscr{MC}(\mathfrak{g}, A)$ for $\mathscr{MC}(\mathfrak{g} \otimes A)$.

The notion of *homotopy* between two Maurer–Cartan elements in an augmented dg algebra $\mathfrak{g}$ can be treated in the same way as for dg Lie algebras, with an appropriate analogue of the Schlessinger–Stasheff, see [CHL21, Theorem 4.1] where this approach is carried out in the smooth context. We will now describe a simple alternative way, that has the added advantage of not requiring that k has characteristic zero.

Consider the dg algebra $\mathfrak{I}$ spanned by two vectors a, b in degree 0 and one vector c in degree 1. The differential is given by

$$d(a) = c, \quad d(b) = -c, \quad d(c) = 0$$

and the algebra structure is specified by

$$a^2 = a, \quad b^2 = b, \quad ca = c, \quad bc = c, \quad ab = ac = ba = cb = c^2 = 0,$$

with unit element $1 = a + b$. This is the cochain algebra on the standard cellular interval with two 0-cells corresponding to the endpoints and one 1-cell. The dg algebra $\mathfrak{g} \otimes \mathfrak{I}$ is a path object for non-unital dg algebra $\mathfrak{g}$. There are two 'evaluation' maps $p_1, p_2 \colon \mathfrak{I} \to k$ so that $p_1(a) = 1$, $p_1(b) = p_1(c) = 0$ and $p_2(b) = 1$, $p_2(a) = p_2(c) = 0$, and these induce the corresponding evaluation maps $\mathfrak{g} \otimes \mathfrak{I} \to \mathfrak{g}$ required in the definition of the path object.

Definition 3.2.10. Let $\mathfrak{g}$ be a non-unital dg algebra. Then two Maurer–Cartan elements $x, y \in \mathfrak{g}$ are *homotopic* if there exists $z \in \mathrm{MC}(\mathfrak{g} \otimes \mathfrak{I})$ such that $(1 \otimes p_1)(z) = x$ and $(1 \otimes p_2)(z) = y$.

We have the following analogue of the Schlessinger–Stasheff theorem.

Theorem 3.2.11. *Let $\mathfrak{g}$ be as in Definition 3.2.10. Then two Maurer–Cartan elements in $\mathfrak{g}$ are homotopic if and only if they are gauge equivalent. In particular, the relation of homotopy on $\mathrm{MC}(\mathfrak{g})$ is an equivalence relation.*

Proof. Any element $z \in \mathfrak{g} \otimes \mathfrak{I} \cong \mathfrak{I} \otimes \mathfrak{g}$ may be written uniquely as $z = a \otimes z_1 + b \otimes z_2 + c \otimes h$ with $z_1, z_2 \in \mathfrak{g}^1$, $h \in \mathfrak{g}^0$. The Maurer–Cartan equation for z is equivalent to z_1 and z_2 being Maurer–Cartan elements such that $d(h) = (1+h)z_1 - z_2(1+h)$ is inside $\mathfrak{g}_e$. Since $1 + h$ is invertible then the latter equation could be rewritten as $z_2 = (1+h)z_1(1+h)^{-1} - d(1+h)(1+h)^{-1}$ so z_1 and z_2 are gauge equivalent. □

3.3. Koszul duality

3.3.1. Quillen equivalence between DGLA and $\mathsf{pcCDGA}^{\mathsf{op}}_{\mathsf{loc}}$. We now explain a Quillen equivalence between DGLA and $\mathsf{pcCDGA}^{\mathsf{op}}_{\mathsf{loc}}$ due to Hinich [Hin01], also called *Koszul duality*, which is at the heart of the modern approach to deformation theory. We assume that the ground field k has characteristic zero. A similar approach works for algebras and (suitably defined) local pseudocompact algebras over a pair of Koszul dual operads; we will not pursue this in full generality but consider, later on, an associative analogue of this story.

Any local augmented pseudocompact commutative dg algebra A with augmentation ideal $I(A)$ determines a dg Lie algebra as follows.

Definition 3.3.1. Let $A \in \mathsf{pcCDGA}_{\mathsf{loc}}$ and set $\mathrm{Harr}(A)$ to be the dg Lie algebra whose underlying space is the free Lie algebra on $\Sigma^{-1}I(A)^*$ and the differential d is defined as $d = d_I + d_{II}$; here d_I is induced by the internal differential on $I(A)$ and d_{II} is determined by its restriction onto $\Sigma^{-1}I(A)^*$ which is in turn induced by the product map $I(A) \otimes I(A) \to I(A)$.

Remark 3.3.2. Note that since $I(A)$ is pseudocompact, its dual $I(A)^*$ is discrete and thus, the dg Lie algebra $\mathrm{Harr}(A)$ is a conventional dg Lie algebra (with no topology). The construction $\mathrm{Harr}(A)$ is the continuous version of the Harrison complex associated with a commutative dg algebra.

Similarly, any dg Lie algebra determines a local pseudocompact commutative dg algebra as follows.

Definition 3.3.3. For a dg Lie algebra $\mathfrak{g}$ set $\mathrm{CE}(\mathfrak{g}) = \widehat{S}\Sigma^{-1}\mathfrak{g}^*$, the completed symmetric algebra on $\Sigma^{-1}\mathfrak{g}^*$. The differential d on $\mathrm{CE}(\mathfrak{g})$ is defined as $d = d_I + d_{II}$; here d_I is induced by the internal differential on $\mathfrak{g}$ and d_{II} is determined by its restriction onto $\Sigma^{-1}\mathfrak{g}^*$ which is in turn induced by the bracket map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$.

The following result holds.

Proposition 3.3.4. *The functors* $\mathrm{Harr}\colon \mathsf{pcCDGA}^{\mathsf{op}}_{\mathsf{loc}} \rightleftarrows \mathsf{DGLA} : \mathrm{CE}$ *form an adjoint pair.*

Proof. We need to show that for $A \in \mathsf{pcCDGA}_{\mathsf{loc}}$ and $\mathfrak{g} \in \mathsf{DGLA}$ there are natural isomorphisms

$$\mathrm{Hom}_{\mathsf{DGLA}}(\mathrm{Harr}(A), \mathfrak{g}) \cong \mathrm{MC}(\mathfrak{g} \otimes A) \cong \mathrm{Hom}_{\mathsf{pcCDGA}_{\mathsf{loc}}}(\mathrm{CE}(\mathfrak{g}), A).$$

We prove the right-hand isomorphism, following [CL11, Proposition 2.2]. Forgetting the differentials, a map $f\colon (\widehat{S}\Sigma^{-1}\mathfrak{g}^*, d) \to (A, d_A)$ in $\mathsf{pcCDGA_{loc}}$ is equivalent to a linear map $\Sigma^{-1}\mathfrak{g}^* \to A$ by freeness of $\widehat{S}$, which is equivalently an element $x_f \in (\mathfrak{g} \otimes A)^1$.

Finally we show that $x_f \in \mathrm{MC}(\mathfrak{g}\otimes A)$ if and only if f commutes with differentials, which says that there are commutative squares

$$\begin{array}{ccccc} \Sigma^{-1}\mathfrak{g}^* & \xrightarrow{d_n} & (\Sigma^{-1}\mathfrak{g}^*)^{\otimes n} & \hookrightarrow & S\Sigma^{-1}\mathfrak{g}^* \\ {\scriptstyle f|_{\Sigma^{-1}\mathfrak{g}^*}}\downarrow & & & & \downarrow{\scriptstyle f} \\ A & & \xrightarrow{\quad d_A \quad} & & A \end{array}$$

To dualize this square, we note that there is a canonical map

$$((\Sigma^{-1}\mathfrak{g}^*)^{\otimes n})^* = ((\Sigma^{-1}\mathfrak{g}^*)^{\otimes n}_{\mathbb{S}_n})^* \longleftarrow ((\Sigma^{-1}\mathfrak{g})^{\otimes n})^{\mathbb{S}_n},$$

given by identifying invariants with coinvariants, as in (2.1.1). Hence the square commuting says $\frac{1}{n!}d_n^* f^* = d_A^* f^*|_{\Sigma^{-1}\mathfrak{g}^*}$, giving the Maurer–Cartan equation for $x_f \in (\mathfrak{g} \otimes A)^1$. The other bijection is proved similarly, using that Harr is defined freely. □

The category $\mathsf{pcCDGA_{loc}}$ has the structure of a model category.

Definition 3.3.5. A morphism $f\colon A \to B$ in $\mathsf{pcCDGA_{loc}}$ is called

(1) a *weak equivalence* if $\mathrm{Harr}(f)\colon \mathrm{Harr}(B) \to \mathrm{Harr}(A)$ is a quasi-isomorphism of dg Lie algebras;
(2) a *fibration* if f is surjective;
(3) a *cofibration* if f has the LLP with respect to all acyclic fibrations.

Theorem 3.3.6. *The category* $\mathsf{pcCDGA_{loc}}$ *together with the classes of fibrations, cofibrations and weak equivalences is a model category. Moreover, the adjoint pair of functors* $(\mathrm{Harr}, \mathrm{CE})$ *is a Quillen equivalence between* $\mathsf{pcCDGA_{loc}^{op}}$ *and* DGLA.

Proof. See [Hin01]. □

Remark 3.3.7. By definition, all objects in the $\mathsf{pcCDGA_{loc}}$ are fibrant, so by Proposition 2.3.23 it is right proper.

The notion of the Maurer–Cartan moduli set has a natural interpretation in terms of model structures on DGLA and $\mathsf{pcCDGA_{loc}}$.

THEOREM 3.3.8. *Let $\mathfrak{g}$ be a dg Lie algebra and A be a local pseudocompact dg algebra. Then there are the following isomorphisms, natural in both variables:*

$$[\mathrm{Harr}(A), \mathfrak{g}] \cong \mathscr{MC}(\mathfrak{g}, A) \cong [\mathrm{CE}(\mathfrak{g}), A].$$

PROOF. The bijection $[\mathrm{Harr}(A), \mathfrak{g}] \cong [\mathrm{CE}(\mathfrak{g}), A]$ follows from the adjunction $(\mathrm{Harr}, \mathrm{CE})$ on the level of homotopy categories. Since $\mathrm{Harr}(A)$ is a cofibrant dg Lie algebra, $[\mathrm{Harr}(A), \mathfrak{g}]$ can be identified with Sullivan homotopy classes of maps $\mathrm{Harr}(A) \to \mathfrak{g}$ (choosing $\mathfrak{g}[t, dt]$ as a path object for $\mathfrak{g}$) and the latter set can be identified, by Theorem 3.2.5 with $\mathrm{MC}(\mathfrak{g} \otimes A)$ modulo gauge equivalence, i.e. with $\mathscr{MC}(\mathfrak{g}, A)$. Note that $\mathfrak{g} \otimes A$ may not be nilpotent, so we need a pro-nilpotent version of Theorem 3.2.5. □

REMARK 3.3.9. A weak equivalence in $\mathsf{pcCDGA}_{\mathsf{loc}}$ is *not* the same as a quasi-isomorphism. Indeed, let $\mathfrak{g}$ be the ordinary Lie algebra $\mathfrak{sl}_2(k)$. It is well-known that the Chevalley-Eilenberg cohomology of $\mathfrak{sl}_2(k)$ is $\Lambda(x)$, the exterior algebra on one generator x in degree 3 and it follows that $\mathrm{CE}(\mathfrak{g})$ is formal, i.e. quasi-isomorphic to its own cohomology. However, $\mathrm{CE}(\mathfrak{g})$ is not weakly equivalent to $\Lambda(x)$, for if it were, then the dg Lie algebra $\mathrm{Harr}(\mathrm{CE}(\mathfrak{g}))$ would be on the one hand, quasi-isomorphic to $\mathfrak{g}$ by Theorem 3.3.6, and on the other, to $\mathrm{Harr}(\Lambda(x))$. But $\mathrm{Harr}(\Lambda(x))$ is isomorphic to the abelian Lie algebra with one basis vector in degree 2 and it is, of course, not quasi-isomorphic to $\mathfrak{g} = \mathfrak{sl}_2(k)$. In fact, a weak equivalence in $\mathsf{pcCDGA}_{\mathsf{loc}}$ is that of a *filtered* quasi-isomorphism and it is finer that a quasi-isomorphism: every weak equivalence of local pseudocompact commutative dg algebras is a quasi-isomorphism but not vice-versa.

PROPOSITION 3.3.10. *The category* $\mathsf{pcCDGA}_{\mathsf{loc}}^{\mathsf{op}}$ *is compactly generated.*

PROOF. Let us denote by $X_n, n \in \mathbb{Z}$ the commutative algebra $k \oplus \Sigma^n k$ where $\Sigma^n k$ has zero multiplication. We claim that the set $\{X_n, n \in \mathbb{Z}\}$ forms a set of compact generators for $\mathsf{pcCDGA}^{\mathsf{op}}$. To see that note that under the Quillen equivalence of Theorem 3.3.6, the algebra X_n corresponds to the free Lie algebra on one generator in degree $n - 1$. These free Lie algebras clearly form a set of compact generators for dg Lie algebras so the conclusion follows. □

3.3.2. Quillen equivalence between DGA^* and $\mathsf{pcDGA}_{\mathsf{loc}}^{\mathsf{op}}$. We now explain an associative analogue of the picture of Koszul duality from the previous section, where $\mathsf{pcCDGA}^{\mathsf{op}}_{\mathsf{loc}}$ is replaced with $\mathsf{pcDGA}_{\mathsf{loc}}$ (i.e. the commutativity is dropped) and DGLA is replaced with DGA^*, the category of *augmented* dg algebras, cf. [Pos11]. Here we do not insist that the ground field k has characteristic zero.

Any local augmented pseudocompact dg algebra A with augmentation ideal $I(A)$ determines an augmented dg algebra as follows.

DEFINITION 3.3.11. For $A \in \mathsf{pcDGA}_{\mathsf{loc}}$ set $\mathrm{Cobar}(A) = T\Sigma^{-1}I(A)^*$, the uncompleted tensor algebra on the discrete vector space $\Sigma^{-1}I(A)^*$. The differential d on $\mathrm{Cobar}(A)$ is defined as $d = d_I + d_{II}$; here d_I is induced by the internal differential on $I(A)$ and d_{II} is determined by its restriction onto $\Sigma^{-1}I(A)^*$ which is in turn induced by the product map $I(A) \otimes I(A) \to I(A)$.

Similarly, any augmented dg algebra $\mathfrak{g}$ with augmentation ideal $\mathfrak{I}(\mathfrak{g})$ determines a local pseudocompact dg algebra as follows.

DEFINITION 3.3.12. For $\mathfrak{g} \in \mathsf{DGA}^*$ set $\mathrm{Bar}(\mathfrak{g}) = \widehat{T}\Sigma^{-1}\mathfrak{I}(\mathfrak{g})^*$, the completed tensor algebra on $\Sigma^{-1}\mathfrak{I}(\mathfrak{g})^*$. The differential d on $\mathrm{Bar}(\mathfrak{g})$ is defined as $d = d_I + d_{II}$; here d_I is induced by the internal differential on $\mathfrak{g}$ and d_{II} is determined by its restriction onto $\Sigma^{-1}\mathfrak{I}(\mathfrak{g})^*$ which is in turn induced by the product map $\mathfrak{I}(\mathfrak{g}) \otimes \mathfrak{I}(\mathfrak{g}) \to \mathfrak{I}(\mathfrak{g})$.

REMARK 3.3.13. The construction $\mathrm{Bar}(\mathfrak{g})$ is commonly referred to as the bar-construction of the dg algebra $\mathfrak{g}$; its cohomology computes $\mathrm{Ext}_{\mathfrak{g}}(k, k)$. Similarly, $\mathrm{Cobar}(A)$ is the cobar-construction of the pseudocompact dg algebra A (or its dual dg coalgebra).

The following result holds.

PROPOSITION 3.3.14. *The functors* $\mathrm{Cobar}\colon \mathsf{pcDGA}^{\mathsf{op}}_{\mathsf{loc}} \leftrightarrows \mathsf{DGA}^* : \mathrm{Bar}$ *form an adjoint pair.*

PROOF. Apply the same argument as for Proposition 3.3.4 to show that for any $A \in \mathsf{pcDGA}_{\mathsf{loc}}$ and $\mathfrak{g} \in \mathsf{DGA}^*$ there are natural isomorphisms

$$\mathrm{Hom}_{\mathsf{DGA}^*}(\mathrm{Cobar}(A), \mathfrak{g}) \cong \mathrm{MC}(\mathfrak{I}(\mathfrak{g}) \otimes A) \cong \mathrm{Hom}_{\mathsf{pcDGA}_{\mathsf{loc}}}(\mathrm{Bar}(\mathfrak{g}), A).$$

The proof is easier than the proof of Proposition 3.3.4, as there is no need to identify invariants with coinvariants. As a result, no factorials appear in the resulting Maurer–Cartan condition. □

The category $\mathsf{pcDGA}_{\mathsf{loc}}$ has the structure of a model category.

DEFINITION 3.3.15. A morphism $f\colon A \to B$ in $\mathsf{pcDGA}_{\mathsf{loc}}$ is called

(1) a *weak equivalence* if $\mathrm{Cobar}(f)\colon\ \mathrm{Cobar}(B) \to \mathrm{Cobar}(A)$ is a quasi-isomorphism of dg algebras;
(2) a *fibration* if f is surjective;
(3) a *cofibration* if f has the LLP with respect to all acyclic fibrations.

THEOREM 3.3.16. *The category* $\mathsf{pcDGA}_{\mathsf{loc}}$ *together with the classes of fibrations, cofibrations and weak equivalences is a model category. Moreover, the adjoint pair of functors* $(\mathrm{Cobar}, \mathrm{Bar})$ *is a Quillen equivalence between* $\mathsf{pcDGA}_{\mathsf{loc}}^{\mathsf{op}}$ *and* DGA^*.

PROOF. See [Pos11]. □

REMARK 3.3.17. By definition, all objects in the $\mathsf{pcDGA}_{\mathsf{loc}}$ are fibrant, so by Proposition 2.3.23 it is right proper.

THEOREM 3.3.18. *There are the following isomorphisms, natural in both variables:*

$$[\mathrm{Cobar}(A), \mathfrak{g}]_{\mathsf{DGA}^*} \cong \mathscr{MC}(\mathfrak{I}(\mathfrak{g}), A) \cong [\mathrm{Bar}(\mathfrak{g}), A]_{\mathsf{pcDGA}_{\mathsf{loc}}}.$$

PROOF. The proof is the same as that of Theorem 3.3.8 with $\mathrm{Harr}(A)$ and $\mathrm{CE}(\mathfrak{g})$ replaced by $\mathrm{Cobar}(A)$ and $\mathrm{Bar}(\mathfrak{g})$ respectively. The only difference is that we choose the smaller path object $\mathfrak{I}(\mathfrak{g}) \otimes \mathfrak{I}$ for $\mathfrak{I}(\mathfrak{g})$ and apply Theorem 3.2.11 to identify homotopy classes of maps $\mathrm{Cobar}(A) \to \mathfrak{g}$ with $\mathscr{MC}(\mathfrak{I}(\mathfrak{g}), A)$. □

REMARK 3.3.19. A weak equivalence in $\mathsf{pcDGA}_{\mathsf{loc}}$ is *not* the same as a quasi-isomorphism. Indeed, let $\mathfrak{g}$ be ordinary associative algebra $k \times k$, the product of two copies of k. Then $\mathrm{Bar}(\mathfrak{g})$ is easily seen to be the dual to the bar-resolution of the algebra k, in particular it is quasi-isomorphic to k. If it were weakly equivalent to k in $\mathsf{pcDGA}_{\mathsf{loc}}$ then $\mathrm{Cobar}(\mathrm{Bar}(\mathfrak{g}))$ would be, on the one hand, quasi-isomorphic to $\mathfrak{g} \cong k \times k$ and, on the other, to $\mathrm{Cobar}(k) \cong k$ giving a contradiction. In fact, a weak equivalence in $\mathsf{pcDGA}_{\mathsf{loc}}$ is that of a filtered quasi-isomorphism and it is finer that a quasi-isomorphism: every weak equivalence of local pseudocompact dg algebras is a quasi-isomorphism but not vice-versa.

PROPOSITION 3.3.20. *The category* $\mathsf{pcDGA}_{\mathsf{loc}}^{\mathsf{op}}$ *is compactly generated.*

PROOF. The argument is the same as in Proposition 3.3.10, using Theorem 3.3.16 in place of Theorem 3.3.6. □

3.3.3. Relationship between two types of Koszul duality. We will now discuss how the associative Koszul duality is related to the Lie-commutative one.

Given a dg Lie algebra $\mathfrak{g}$, its universal enveloping algebra $U\mathfrak{g}$ is a dg algebra; this determines a functor $\mathsf{DGLA} \to \mathsf{DGA}^*$ that is left adjoint to the functor Lie taking an associative augmented dg algebra to the commutator dg Lie algebra of its augmentation ideal. Similarly the forgetful functor $\mathrm{Ass}\colon \mathsf{pcCDGA}_{\mathsf{loc}} \to \mathsf{pcDGA}_{\mathsf{loc}}$ is right adjoint to the *abelianization* functor $\mathrm{Ab}\colon \mathsf{pcDGA}_{\mathsf{loc}} \to \mathsf{pcCDGA}_{\mathsf{loc}}$, associating to an associative pseudocompact dg algebra $\mathfrak{g}$ its quotient by the ideal topologically generated by (graded) commutators in $\mathfrak{g}$. It is clear that both are in fact Quillen adjunctions.

Proposition 3.3.21. *The following diagrams of model categories and Quillen functors between them is commutative in the sense that there is a functor isomorphism* $U \circ \mathrm{Harr} \cong \mathrm{Cobar} \circ \mathrm{Ass}$ *and* $\mathrm{CE} \circ \mathrm{Lie} \cong \mathrm{Ab} \circ \mathrm{Bar}$.

$$
\begin{array}{ccc}
\mathsf{DGA}^* & \xleftarrow{U} & \mathsf{DGLA} \\
{\scriptstyle\mathrm{Cobar}}\uparrow & & \uparrow{\scriptstyle\mathrm{Harr}} \\
\mathsf{pcDGA}^{\mathsf{op}}_{\mathsf{loc}} & \xleftarrow{\mathrm{Ass}} & \mathsf{pcCDGA}^{\mathsf{op}}_{\mathsf{loc}}
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{DGA}^* & \xrightarrow{\mathrm{Lie}} & \mathsf{DGLA} \\
{\scriptstyle\mathrm{Bar}}\downarrow & & \downarrow{\scriptstyle\mathrm{CE}} \\
\mathsf{pcDGA}^{\mathsf{op}}_{\mathsf{loc}} & \xrightarrow{\mathrm{Ab}} & \mathsf{pcCDGA}^{\mathsf{op}}_{\mathsf{loc}}
\end{array}
$$

Proof. Straightforward unravelling of the definitions. □

3.4. Main theorems

3.4.1. Maurer–Cartan elements and the deformation functor based on a dg Lie algebra. Any dg Lie algebra $\mathfrak{g}$ determines a deformation functor $\mathrm{Def}_{\mathfrak{g}}\colon A \mapsto \mathrm{Def}_{\mathfrak{g}}(A) = \mathscr{MC}(\mathfrak{g}, A)$ where A is a local pseudocompact commutative dg algebra. Thus, $\mathrm{Def}_{\mathfrak{g}}$ is a set-valued functor on $\mathsf{pcCDGA}_{\mathsf{loc}}$. This (extended) deformation functor has the following homotopy invariance property.

Theorem 3.4.1. *Let* $\mathfrak{g}$ *be a dg Lie algebra.*

(1) *If* $A \to B$ *is a weak equivalence in* $\mathsf{pcCDGA}_{\mathsf{loc}}$ *then the induced map* $\mathrm{Def}_{\mathfrak{g}}(A) \to \mathrm{Def}_{\mathfrak{g}}(B)$ *is an isomorphism. Therefore* $\mathrm{Def}_{\mathfrak{g}}$ *descends to a set-valued functor on* $\mathsf{Ho}(\mathsf{pcCDGA}_{\mathsf{loc}})$ *that will be denoted by the same symbol.*

(2) *If* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *are two quasi-isomorphic dg Lie algebras, then the functors* $\mathrm{Def}_{\mathfrak{g}}$ *and* $\mathrm{Def}_{\mathfrak{g}'}$ *are isomorphic.*

Proof. This follows from Theorem 3.3.8. □

Theorem 3.4.2. *The set-valued functor* $\mathrm{Def}_{\mathfrak{g}}$ *on* $\mathsf{Ho}(\mathsf{pcCDGA_{loc}})$ *is representable by the local pseudocompact commutative dg algebra* $\mathrm{CE}(\mathfrak{g})$*. Conversely, any functor on* $\mathsf{Ho}(\mathsf{pcCDGA_{loc}})$ *that is homotopy representable by a local pseudocompact commutative dg algebra* A *is isomorphic to the functor* $\mathrm{Def}_{\mathrm{Harr}(A)}$*.*

Proof. By Theorem 3.3.8 we have $\mathrm{Def}_{\mathfrak{g}}(A) = \mathscr{MC}(\mathfrak{g}, A) \cong [\mathrm{CE}(\mathfrak{g}), A]$, which means that $\mathrm{Def}_{\mathfrak{g}}$ is representable by $\mathrm{CE}(\mathfrak{g})$. Conversely, given a functor F on $\mathsf{pcCDGA_{loc}}$ representable by a local pseudocompact dg algebra A we have for $B \in \mathsf{pcCDGA_{loc}}$:

$$\begin{aligned} F(B) &= [B, A] \\ &\cong [\mathrm{CE}(\mathrm{Harr}(A)), B] \\ &\cong \mathscr{MC}(\mathrm{Harr}(A), B) \\ &\cong \mathrm{Def}_{\mathrm{Harr}(A)}(B) \end{aligned}$$

as required. □

3.4.2. Finding a dg Lie algebra associated with a deformation functor. We will now formulate the necessary and sufficient conditions on a homotopy invariant functor on pcCDGA ensuring that it is representable (and thus, 'controlled' by a dg Lie algebra).

Theorem 3.4.3. *Let* F *be a set-valued functor on* $\mathsf{pcCDGA_{loc}}$ *such that:*

(1) *F is homotopy invariant: it takes weak equivalences in $\mathsf{pcCDGA_{loc}}$ to bijections of sets.*
(2) *F is normalized: $F(k)$ is a one-element set.*
(3) *F takes arbitrary products in $\mathsf{pcCDGA_{loc}}$ into products of sets.*
(4) *For any diagram in $\mathsf{pcCDGA_{loc}}$ of the form $B \to A \leftarrow C$ where $A \leftarrow C$ is surjective, the natural map $F(B \times_A C) \to F(B) \times_{F(A)} F(C)$ is surjective.*

Then F is homotopy representable, i.e. there exists $X \in \mathsf{pcCDGA_{loc}}$ such that for any $Y \in \mathsf{pcCDGA_{loc}}$ there is a natural isomorphism $F(Y) \cong [X, Y]$.

Proof. This follows from Brown representability, Theorem 3.1.4, taking into account that the model category $\mathsf{pcCDGA_{loc}^{op}}$ is compactly generated, cf. Proposition 3.3.10. □

REMARK 3.4.4. One can consider deformation functors with values in simplicial sets, rather than sets. This is the approach taken in [Lur, Pri10]. There is a version of the representability theorem in this setting.

3.4.3. Associative deformation theory. Any augmented dg algebra $\mathfrak{g}$ over a field k of arbitrary characteristic determines a deformation functor $\mathrm{Def}_{\mathfrak{g}}\colon A \mapsto \mathrm{Def}_{\mathfrak{g}}(A) = \mathscr{MC}(\mathfrak{g}, A)$ where A is a local pseudocompact associative dg algebra. Thus, $\mathrm{Def}_{\mathfrak{g}}$ is a set-valued functor on $\mathsf{pcDGA}_{\mathsf{loc}}$. This (extended) deformation functor has the following homotopy invariance property.

THEOREM 3.4.5. *Let $\mathfrak{g}$ be an augmented dg algebra.*

(1) *If $A \to B$ is a weak equivalence in $\mathsf{pcDGA}_{\mathsf{loc}}$ then the induced map $\mathrm{Def}_{\mathfrak{g}}(A) \to \mathrm{Def}_{\mathfrak{g}}(B)$ is an isomorphism. Therefore $\mathrm{Def}_{\mathfrak{g}}$ descends to a set-valued functor on $\mathsf{Ho}(\mathsf{pcDGA}_{\mathsf{loc}})$ that will be denoted by the same symbol.*
(2) *If $\mathfrak{g}$ and $\mathfrak{g}'$ are two quasi-isomorphic dg algebras, then the functors $\mathrm{Def}_{\mathfrak{g}}$ and $\mathrm{Def}_{\mathfrak{g}'}$ are isomorphic.*

PROOF. This follows from Theorem 3.3.18. □

THEOREM 3.4.6. *The set-valued functor $\mathrm{Def}_{\mathfrak{g}}$ on $\mathsf{Ho}(\mathsf{pcDGA}_{\mathsf{loc}})$ is representable by the local pseudocompact dg algebra $\mathrm{Bar}(\mathfrak{g})$. Conversely, any functor on $\mathsf{Ho}(\mathsf{pcDGA}_{\mathsf{loc}})$ that is homotopy representable by a local pseudocompact dg algebra A is isomorphic to the functor $\mathrm{Def}_{\mathrm{Cobar}(A)}$.*

PROOF. The proof is the same as that of Theorem 3.4.2, applying Theorem 3.3.18 instead of Theorem 3.3.8. □

3.4.4. Finding a dg algebra associated with a deformation functor. We will now formulate the necessary and sufficient conditions on a homotopy invariant functor on $\mathsf{pcDGA}_{\mathsf{loc}}$ ensuring that it is representable (and thus, 'controlled' by an augmented (or, equivalently, non-unital) dg algebra).

THEOREM 3.4.7. *Let F be a set-valued functor on $\mathsf{pcDGA}_{\mathsf{loc}}$ such that:*

(1) *F is homotopy invariant: it takes weak equivalences in $\mathsf{pcDGA}_{\mathsf{loc}}$ to bijections of sets;*
(2) *F is normalized: $F(k)$ is a one-element set.*

(3) *F takes arbitrary products in $\mathsf{pcDGA}_{\mathsf{loc}}$ into products of sets.*

(4) *For any diagram in $\mathsf{pcDGA}_{\mathsf{loc}}$ of the form $B \to A \leftarrow C$ where $A \leftarrow C$ is surjective, the natural map $F(B \times_A C) \to F(B) \times_{F(A)} F(C)$ is surjective.*

Then F is homotopy representable, i.e. there exists $X \in \mathsf{pcDGA}_{\mathsf{loc}}$ such that for any $Y \in \mathsf{pcCDGA}_{\mathsf{loc}}$ there is a natural isomorphism $F(Y) \cong [X, Y]$.

PROOF. This follows from Brown representability, Theorem 3.1.4, taking into account that the model category $\mathsf{pcDGA}_{\mathsf{loc}}^{\mathsf{op}}$ is compactly generated, cf. Proposition 3.3.10. □

3.4.5. Comparing commutative and associative deformations. Assume now that k has characteristic zero. Any set-valued functor F on $\mathsf{pcDGA}_{\mathsf{loc}}$ determines by restriction a functor on $\mathsf{pcCDGA}_{\mathsf{loc}}$ and so it makes sense to ask whether an associative deformation functor $\mathrm{Def}_{\mathfrak{g}}$ for $\mathfrak{g} \in \mathsf{DGA}^*$ restricts to a deformation functor on $\mathsf{pcCDGA}_{\mathsf{loc}}$. The following results answer this question.

THEOREM 3.4.8. *Let $\mathfrak{g}$ be a dg algebra. Then the deformation functor $\mathrm{Def}_{\mathfrak{g}}$ on $\mathsf{pcDGA}_{\mathsf{loc}}$ restricts to the deformation functor $\mathrm{Def}_{\mathrm{Lie}(\mathfrak{g})}$ on $\mathsf{pcCDGA}_{\mathsf{loc}}$.*

PROOF. We know by Theorem 3.4.6 that $\mathrm{Def}_{\mathfrak{g}}$ is represented by a dg algebra $\mathrm{Bar}(\mathfrak{g})$. Then for $\mathfrak{h} \in \mathsf{pcCDGA}_{\mathsf{loc}}$ we have $\mathrm{Def}_{\mathfrak{g}}(\mathfrak{h}) = [\mathrm{Bar}(\mathfrak{g}), \mathfrak{h}]_{\mathsf{DGA}^*}$ and so by Proposition 3.3.21 and Theorem 3.3.8 we have:

$$
\begin{aligned}
\mathrm{Def}_{\mathfrak{g}}(\mathfrak{h}) &\cong [\mathrm{Ab}(\mathrm{Bar}(\mathfrak{g})), \mathfrak{h}]_{\mathsf{pcCDGA}_{\mathsf{loc}}} \\
&\cong [\mathrm{CE}(\mathrm{Lie}(\mathfrak{g})), \mathfrak{h}]_{\mathsf{pcCDGA}_{\mathsf{loc}}} \\
&\cong \mathscr{MC}(\mathrm{Lie}(\mathfrak{g}), \mathfrak{h}) \\
&\cong \mathrm{Def}_{\mathrm{Lie}(\mathfrak{g})}(\mathfrak{h})
\end{aligned}
$$

as claimed. □

REMARK 3.4.9. As we saw, every deformation functor in characteristic zero is controlled by a dg Lie algebra. On the other hand, not every deformation functor is defined on the category $\mathsf{pcDGA}_{\mathsf{loc}}$ (which would imply that it is controlled by an *associative* dg algebra), in the same way as not every Lie algebra comes from an associative algebra. An interesting example of an associative deformation theory is that of deformations of modules over an associative algebra. Let $\mathfrak{g}$ be an algebra and M be a $\mathfrak{g}$-module. Deformations of M are controlled by the dg algebra

$\mathrm{REnd}(M)$, the derived endomorphism algebra of M viewed as a non-unital algebra (which can be obtained as the ordinary endomorphism algebra of a $\mathfrak{g}$-projective resolution of M). Considered as a functor on $\mathsf{pcCDGA}_{\mathsf{loc}}$, this deformation theory is controlled by $\mathrm{Lie}(\mathrm{REnd}(M))$, the commutator Lie algebra of $\mathrm{REnd}(M)$. More generally, deformations of A_∞-modules over an A_∞-algebra are controlled by a certain non-unital dg algebra, cf. [GLST20a] regarding this example.

CHAPTER 4

Gauge equivalence for complete L_∞-algebras

In this chapter, we turn our attention to the Schlessinger–Stasheff theorem, and show that an analogous statement holds in the generality of L_∞- and A_∞-algebras. As a particularly interesting application, we show that a non-abelian version of the Poincaré lemma holds for differential forms with values in an L_∞-algebra.

Throughout this chapter we assume k is a field of characteristic zero.

4.1. Strongly homotopy algebras

We start by recalling basic facts on L_∞- and A_∞-algebras, which are strongly homotopy versions of dg Lie algebras and dg algebras. Afterwards we give definitions of Maurer–Cartan elements and Sullivan homotopy in this more general setting. All definitions given in this section are standard and agree with those commonly found in the literature, except for the notion of completeness: in particular, the definition given here agrees with [LM15] but not with [BFMT18].

A_∞-algebras, and later L_∞-algebras, were originally defined in [Sta63] and [LS93] respectively as graded vector spaces V together with collections of linear maps $V^{\otimes n} \to V$, $n = 1, 2, \ldots$ satisfying various identities. We choose to give more concise definitions below, which say that an L_∞- or A_∞-structure is determined by a Maurer–Cartan element in a suitable graded Lie algebra, and briefly explain how the two definitions are equivalent.

Definition 4.1.1 (following [HL09]). Let V be a graded vector space.

(1) An L_∞*-structure* on V is a continuous degree 1 derivation m of the complete cdga $\widehat{S}\Sigma^{-1}V^*$, such that $m^2 = 0$ and m has no constant term. The pair (V, m) is called an L_∞*-algebra*, and $(\widehat{S}\Sigma^{-1}V^*, m)$ is called its *representing complete cdga.*

Given two L_∞-algebras (U, m_U) and (V, m_V), an L_∞*-morphism* $U \to V$ is a continuous cdga map $(\widehat{S}\Sigma^{-1}V^*, m_V) \to (\widehat{S}\Sigma^{-1}U^*, m_U)$.

(2) An A_∞*-structure* on V is a continuous degree 1 derivation m of the complete dga $\widehat{T}\Sigma^{-1}V^*$, such that $m^2 = 0$ and m has no constant term. The pair

(V, m) is called an A_∞-*algebra*, and $(\widehat{T}\Sigma^{-1}V^*, m)$ is called its *representing complete dga*.

Given two A_∞-algebras (U, m_U) and (V, m_V), an A_∞-*morphism* $U \to V$ is a continuous dga map $(\widehat{T}\Sigma^{-1}V^*, m_V) \to (\widehat{T}\Sigma^{-1}U^*, m_U)$.

One recovers the standard definition of an L_∞-structure as a sequence of graded maps as follows: By definition, the derivation m is determined by its components $m_i\colon \Sigma^{-1}V^* \to S^i\Sigma^{-1}V^*$, $i \geq 1$. Dualize the components m_i and apply the canonical identification (2.1.1) of $\mathbb{S}_n$-invariants and $\mathbb{S}_n$-coinvariants, to get graded symmetric maps $\ell_i\colon S^i\Sigma V \to \Sigma V$ of degree 1, with $m_i = \frac{1}{i!}\ell_i^*$. The condition $m^2 = 0$ then translates into higher Jacobi identities.

Under the identification $S^i\Sigma V \cong \Lambda^i V$, an L_∞-structure on V is equivalently a sequence of graded antisymmetric brackets $[-, \ldots, -]_i\colon \Lambda^i V \to V$ of degree $2 - i$. For later convenience, we adopt the convention that the graded symmetric and graded antisymmetric operations are related by $\ell_i = \Sigma\,[-, \ldots, -]_i\,(\Sigma^{-1})^{\otimes i}$, so that by the Koszul sign rule,

$$\ell_i(x_1, \ldots, x_i) = (-1)^{\sum_{j=1}^{i-1}(i-j)|x_j|}\,\Sigma\,[\Sigma^{-1}x_1, \ldots, \Sigma^{-1}x_i].$$

Analogously, an A_∞-structure is equivalent to a sequence of graded maps $T^iV \to V$, $i \geq 1$, of degree $2 - i$, satisfying higher associativity identities. Note that factorials do not appear in the A_∞-algebra case, because there is no need to identify invariants and coinvariants.

Remark 4.1.2. Note that a dg Lie algebra structure on V is an L_∞-structure on V where the derivation m is further required to be quadratic, that is, $m = m_1 + m_2$ with $m_1\colon \Sigma^{-1}V^* \to \Sigma^{-1}V^*$ and $m_2\colon \Sigma^{-1}V^* \to \Sigma^{-1}V^*\widehat{\otimes}\Sigma^{-1}V^*$. In this case, m_1 corresponds to the differential of V and m_2 corresponds to the Lie bracket. The condition $m^2 = 0$ then says precisely that the differential of V has square zero, and that the Lie bracket of V satisfies the Jacobi identity. Thus, L_∞-algebras generalize (dg) Lie algebras. Arguing similarly, we also have that A_∞-algebras are a generalization of (dg) algebras.

For our purposes, V will often be a pseudocompact vector space instead of discrete. In this case, an L_∞-structure, A_∞-structure, etc., on V is defined by replacing the complete cdga $\widehat{S}\Sigma^{-1}V^*$ in Definition 4.1.1 with the cdga $S\Sigma^{-1}V^*$, and replacing the complete dga $\widehat{T}\Sigma^{-1}V^*$ with the dga $T\Sigma^{-1}V^*$.

Recall from Example 2.3.4(3) that the categories CDGA and DGA have model structures in which the weak equivalences are quasi-isomorphisms and the fibrations are degreewise surjections. All objects are therefore fibrant. We now give a description of the cofibrant objects.

DEFINITION 4.1.3. A *Sullivan cdga* (resp. *Sullivan dga*) is defined to be a cdga of the form SV (resp. dga of the form TV) such that V is a graded vector space admitting a filtration

$$0 \subseteq V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V, \quad V = \bigcup_{i \geq 0} V_i,$$

that is compatible with the differential d, i.e. $d(V_i) \subseteq SV_{i-1}$ (resp. $d(V_i) \subseteq TV_{i-1}$) for all i.

The cofibrant objects in the model categories of (c)dgas are precisely retracts of Sullivan (c)dgas. A proof appears in, for example, [Pos11, Theorem 9.1] for dgas; the same proof also works for cdgas.

DEFINITION 4.1.4. An L_∞-algebra (resp. A_∞-algebra) V is *complete* if V is pseudocompact and its representing cdga (resp. dga) is cofibrant in the model category of cdgas (resp. dgas).

DEFINITION 4.1.5.

(1) Let (V, m) be a complete L_∞-algebra and A be a cdga. An element $\xi \in V \otimes A$ is *Maurer–Cartan* if it has degree 1 and satisfies the *Maurer–Cartan equation*

$$(\mathrm{id} \otimes d_A)(\xi) + \sum_{i \geq 1} \frac{1}{i!}[\xi, \ldots, \xi]_i^A = 0,$$

where $[-, \ldots, -]_i^A$ is the A-linear extension of $[-, \ldots, -]_i$.

(2) Let (V, m) be a complete A_∞-algebra and A be a cdga. An element $\xi \in V \otimes A$ is *Maurer–Cartan* if it has degree 1 and satisfies the *Maurer–Cartan equation*

$$(\mathrm{id} \otimes d_A)(\xi) + \sum_{i \geq 1} m_i^A(\xi, \ldots, \xi) = 0,$$

where m_i^A is the A-linear extension of m_i.

The set of all Maurer–Cartan elements in $V \otimes A$ is denoted by $\mathrm{MC}(V, A)$. In the case where $A = k$, we write $\mathrm{MC}(V, k)$ simply as $\mathrm{MC}(V)$.

Remark 4.1.6. The completeness condition on the L_∞- and A_∞-algebra ensures that the infinite sums converge in Definition 4.1.5.

Given any cdga A and complete L_∞-algebra V, a Maurer–Cartan element in the L_∞-algebra $V \otimes A$ is represented by a cdga map $S\Sigma^{-1}V^* \to A$. Similarly, given any dga A, a Maurer–Cartan element in the A_∞-algebra $V \otimes A$ is represented by a dga map $T\Sigma^{-1}V^* \to A$. For details see, for example, Proposition 2.2 and Remark 2.3 in [CL11].

Next we consider a notion of homotopy for Maurer–Cartan elements. Let V be an L_∞-algebra or A_∞-algebra. Consider the L_∞-algebra or A_∞-algebra $V[t, dt] := V \otimes k[t, dt]$, where $k[t, dt]$ denotes the free cdga generated by a degree 0 symbol t and a degree 1 symbol dt. Then there are two natural cdga maps $f_0, f_1 \colon k[t, dt] \to k$, sending t to 0 and 1 respectively, and sending dt to 0.

Definition 4.1.7. Let (V, m) be a complete L_∞-algebra or A_∞-algebra, and A be a cdga. Two elements $\xi, \eta \in \mathrm{MC}(V, A)$ are *Sullivan homotopic* if there exists an element $h \in \mathrm{MC}(V, A[t, dt])$ such that $(\mathrm{id} \otimes f_0)(h) = \xi$ and $(\mathrm{id} \otimes f_1)(h) = \eta$.

The following result is well-known; see for example [Laz13].

Proposition 4.1.8. *Let (V, m) be a complete L_∞-algebra or A_∞-algebra, and let A be a cdga. Two Maurer–Cartan elements $\xi, \eta \in \mathrm{MC}(V, A)$ are Sullivan homotopic if and only if their representing (c)dga maps are right homotopic (Definition 2.3.6) in the model category of (c)dgas.*

Proof. This is immediate from regarding $h \in \mathrm{MC}(V \otimes A[t, dt])$ as a cdga map $S\Sigma^{-1}V^* \to A[t, dt]$ or a dga map $T\Sigma^{-1}V^* \to A[t, dt]$. Then

$$A \xrightarrow{\ i\ } A[t, dt] \overset{f_0}{\underset{f_1}{\rightrightarrows}} A$$

is a good path object for A in the model category of (c)dgas; here i denotes the natural inclusion. □

Finally, for the purpose of this chapter it is useful to rewrite the gauge action from Section 3.2 in the following way. Let V be a complete dgla, with differential d and bracket $[-, -]$. In this case, $(V \otimes A)^0$ is a Lie algebra and the *gauge group* G of $V \otimes A$ is defined by exponentiating $(V \otimes A)^0$. That is, G consists of formal symbols $\{e^x : x \in (V \otimes A)^0\}$, with multiplication $e^x e^y := e^{x*y}$ given by the

Baker–Campbell–Hausdorff (BCH) formula, defined by

$$x * y := \log(e^x e^y)$$

using the formal power series $e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\log(1+t) := \sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^n}{n}$. The first few terms are

$$x * y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \frac{1}{12}[y,[y,x]] - \frac{1}{24}[y,[x,[x,y]]] \cdots .$$

DEFINITION 4.1.9. The *gauge action* of G on $\mathrm{MC}(V,A)$ is defined by

$$e^x \cdot \xi = \xi + \sum_{n=1}^{\infty} \frac{(\mathrm{ad}_x)^{n-1}}{n!}(\mathrm{ad}_x \xi - dx). \tag{4.1.1}$$

Two Maurer–Cartan elements $\xi, \eta \in V \otimes A$ are said to be *gauge equivalent* if they lie in the same orbit of the gauge action. We write $\mathscr{MC}(V,A)$ for the quotient of $\mathrm{MC}(V,A)$ by the gauge action.

If V is a complete dga, we say that two Maurer–Cartan elements in $V \otimes A$ are gauge equivalent if they are *gauge equivalent* in the corresponding dgla, taken with the commutator bracket.

REMARK 4.1.10. Completeness of V implies that V is pronilpotent; thus the infinite series in the BCH formula and the above gauge action (4.1.1) converge. Indeed, the ascending filtration on $S\Sigma^{-1}V^*$ corresponds to a descending filtration on V, and the Sullivan condition on the differential of $S\Sigma^{-1}V^*$ corresponds to pronilpotence of V^* with respect to this filtration. This result appears in [Ber15, Theorem 2.3] in the case where V is assumed to have finite type; we can avoid this since we are working with pseudocompact vector spaces.

Write $\mathfrak{g} = V \otimes A$. If V is a complete dgla, then $\mathfrak{g}$ is pronilpotent and the definition above coincides with Proposition 3.2.2. To see this, we apply a similar trick as in the proof of that proposition to first reduce to the case with zero differential: let $\tilde{\mathfrak{g}}$ be the graded Lie algebra $\mathfrak{g} \oplus k \cdot \delta$ with δ in cohomological degree 1 and $[x,\delta] := -dx$, $[\delta,\delta] = 0$. Then two elements $\xi, \eta \in \mathrm{MC}(\mathfrak{g})$ are gauge equivalent if $\xi + \delta, \eta + \delta \in \mathrm{MC}(\tilde{\mathfrak{g}})$ satisfy $\eta + \delta = e^{\mathrm{ad}_x}(\xi + \delta)$. Indeed,

$$\eta + \delta = e^{\mathrm{ad}_x}(\xi) + e^{\mathrm{ad}_x}(\delta) = e^{\mathrm{ad}_x}(\xi) + (e^{\mathrm{ad}_x} - 1)(\delta) + \delta = e^{\mathrm{ad}_x}(\xi) + \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(-dx) + \delta$$

which is the formula (4.1.1). It therefore suffices to check that $e^{\mathrm{ad}_x}(\xi)$ corresponds to the congugation $x \cdot \xi = x\xi x^{-1}$ in the gauge action defined previously. This can be found in, for example, [BFMT20].

Our aim is to establish gauge equivalence as a *left* homotopy in the model category of cdgas. In particular, left and right homotopy coincide when the domain is cofibrant and the codomain is fibrant, so this would prove that Sullivan homotopy and gauge equivalence coincide in the case of dglas. We will see how this interpretation extends to Maurer–Cartan elements in L_∞-algebras.

4.2. Gauge equivalence as a left homotopy of DGLAs

It this section, we show that gauge equivalence for complete dglas coincides with left homotopy between complete dgla morphisms, with respect to the model category structure of [LM15]. An analogous result is proved in [BM13b, BFMT18] for a different model structure. The key to this construction lies in an alternative characterization of Maurer–Cartan elements in V, when V is a dgla, as follows. Let $L(x)$ be the free complete dgla generated by one element x of degree 1, with differential $dx = -\frac{1}{2}[x, x]$. Then there is a correspondence between the set $\mathrm{MC}(V)$ and the set of dgla morphisms $L(x) \to V$. Thus, to describe a left homotopy in a model category of complete dglas we require a cylinder object for $L(x)$; such a cylinder is given by the *Lawrence–Sullivan interval* introduced in [LS14].

The Lawrence–Sullivan interval $\mathfrak{L}$ is the free complete dgla on three generators a, b, z, where $|a| = |b| = 1$, $|z| = 0$, with differential

$$da + \frac{1}{2}[a, a] = 0, \quad db + \frac{1}{2}[b, b] = 0,$$
$$dz = \mathrm{ad}_z(b) + \frac{\mathrm{ad}_z}{e^{\mathrm{ad}_z} - \mathrm{id}}(b - a).$$

That is, the differential d is defined such that a and b are Maurer–Cartan elements in $\mathfrak{L}$, and are gauge equivalent by $a = e^z \cdot b$.

PROPOSITION 4.2.1. *Let* $i_0, i_1 \colon L(x) \to \mathfrak{L}$ *be the natural inclusions and* $p \colon \mathfrak{L} \to L(x)$ *be the natural projection, that is,* $i_0(x) = a$, $i_1(x) = b$ *and* $p(a) = p(b) = x$, $p(z) = 0$. *Then*

$$L(x) \underset{i_1}{\overset{i_0}{\rightrightarrows}} \mathfrak{L} \xrightarrow{\ p\ } L(x)$$

is a good cylinder object for $L(x)$ *in the category of complete dglas, equipped with a model structure in which a morphism* $f \colon (V, d) \to (V', d')$ *is*

(1) *a weak equivalence if* $S\Sigma^{-1}(V')^* \to S\Sigma^{-1}V^*$ *is a weak equivalence in the category of cdgas.*
(2) *a fibration if it is surjective.*

PROOF. See [BFMT18] Corollary 5.3 and Theorem 7.6. ☐

It is then straightforward to show that gauge equivalence corresponds to the notion of a left homotopy in the category of complete dglas.

PROPOSITION 4.2.2. *Let V be a complete dgla. Two Maurer–Cartan elements $\xi, \eta \in V$ are gauge equivalent if and only if there exists a dgla morphism $h\colon \mathfrak{L} \to V$ such that $h(a) = \xi$ and $h(b) = \eta$.*

PROOF. Consider the element $h(z) \in V$. If h is a dgla morphism, then $h(z)$ has degree 0, and $d(h(z)) = h(dz)$, from which a direct computation shows $\xi = e^{h(z)} \cdot \eta$. Conversely, if ξ and η are gauge equivalent by $\xi = e^x \cdot \eta$, then define $h(a) = \xi$, $h(b) = \eta$ and $h(z) = x$. The same computation shows that h is a dgla morphism. ☐

It is natural to ask how this result can be generalized to a notion of homotopy when V is an L_∞-algebra. In [BM13a], two Maurer–Cartan elements $\xi, \eta \in V$ are called *cylinder homotopic* (terminology following [DP16]) if there exists an L_∞-morphism $\mathfrak{L} \to V$ such that $h(a) = \xi$ and $h(b) = \eta$. Then by [BM13a, Proposition 4.5], two Maurer–Cartan elements are cylinder homotopic if and only if they are Sullivan homotopic. The resulting generalization, however, is no longer a left homotopy of complete dglas.

4.3. Left homotopy of Maurer–Cartan elements

A different approach will be taken in this section: loosely, we will work in the Koszul dual picture, and consider Maurer–Cartan elements in L_∞-algebras and A_∞-algebras by their representing (c)dga maps. We define the following notion of homotopy for Maurer–Cartan elements.

DEFINITION 4.3.1.

(1) Let V be a complete L_∞-algebra and A be a cdga. Two Maurer–Cartan elements $\xi, \eta \in V \otimes A$ are *left homotopic* if their representing cdga maps $S\Sigma^{-1}V^* \to A$ are left homotopic in the model category of cdgas.

(2) Let V be a complete A_∞-algebra and A be a cdga. Two Maurer–Cartan elements $\xi, \eta \in V \otimes A$ are *left homotopic* if their representing dga maps $T\Sigma^{-1}V^* \to A$ are left homotopic in the model category of dgas.

4.3.1. The cylinder object for (c)dgas. We recall a cylinder object for cdgas constructed in [FOT08, Section 2.2]. Given a cofibrant cdga of the form (SV, d), its *cylinder* $C(SV)$ is defined to be the cdga $(S(V \oplus \bar{V} \oplus \widehat{V}), D)$, where $\bar{V} \cong \Sigma V$ and $\widehat{V} \cong V$, and the differential D is defined by

$$D(v) = dv, \; D(\bar{v}) = \widehat{v}, \; D(\widehat{v}) = 0.$$

We also define a degree -1 derivation s on $C(SV)$ by

$$s(v) = \bar{v}, \; s(\bar{v}) = s(\widehat{v}) = 0.$$

Then $\theta := [s, D] = sD + Ds$ is a derivation of degree 0, so $e^\theta = \sum_{n=0}^{\infty} \theta^n/n!$ is an automorphism of $C(SV)$. Explicitly,

$$\theta(v) = sdv + \widehat{v}, \; \theta(\bar{v}) = \theta(\widehat{v}) = 0,$$

and inductively, $\theta^n(v) = (sD)^n(v)$ for $n \geq 2$ as $s^2 = 0$. Since SV is a Sullivan cdga, $\theta^N(v) = 0$ for some N, and hence we have a convergent series

$$e^\theta(v) = v + \widehat{v} + \sum_{n=1}^{\infty} \frac{(sD)^n(v)}{n!}, \; e^\theta(\bar{v}) = \bar{v}, \; e^\theta(\widehat{v}) = \widehat{v}. \tag{4.3.1}$$

Analogously, given a cofibrant dga of the form (TV, d), its *cylinder* $C(TV)$ is defined to be the dga $(T(V \oplus \bar{V} \oplus \widehat{V}), D)$, with D and e^θ defined as above. This is a different cylinder to the one constructed by [BL77].

PROPOSITION 4.3.2. *Let (SV, d) be a cofibrant cdga. Let $i\colon SV \to C(SV)$ be the natural inclusion and $p\colon C(SV) \to SV$ be the natural projection, that is, $i(v) = v$ and $p(v) = v$, $p(\bar{v}) = p(\widehat{v}) = 0$. Then*

$$SV \underset{e^\theta \circ i}{\overset{i}{\rightrightarrows}} C(SV) \xrightarrow{\;p\;} SV$$

is a good cylinder object for (SV, d) in the model category of cdgas. Analogously, if (TV, d) is a cofibrant dga, then

$$TV \underset{e^\theta \circ i}{\overset{i}{\rightrightarrows}} C(TV) = (T(V \oplus \bar{V} \oplus \widehat{V}), D) \xrightarrow{\;p\;} TV$$

is a good cylinder object for (TV, d) in the model category of dgas.

Since $C(SV)$ and $C(TV)$ are good cylinder objects, ξ and η are left homotopic if and only if there exists a cdga morphism $H\colon C(SV) \to k$ or dga morphism $H\colon C(TV) \to k$ that is a left homotopy between their representing (c)dga maps.

4.3.2. Left homotopy in (c)dgas. From now on, let V be a complete L_∞-algebra; everything we say will have an obvious analogue for complete A_∞-algebras. Consider the vector space $V \oplus \bar{V} \oplus \widehat{V}$, where $\bar{V} \cong \Sigma^{-1}V$ and $\widehat{V} \cong V$. This is a complete L_∞-algebra with differential

$$d(\xi) = d\xi,\ d(\bar{\xi}) = 0,\ d(\widehat{\xi}\,) = \bar{\xi}, \tag{4.3.2}$$

and all brackets defined as 0 on the second and third components. Then the representing cdga of $V \oplus \bar{V} \oplus \widehat{V}$ is isomorphic to $C(SU)$, where $U = \Sigma^{-1}V^*$, $\bar{U} = \Sigma^{-1}\bar{V}^* \cong \Sigma U$ and $\widehat{U} \cong U$, with differential D as in Section 4.3.1.

Note that an element in $V \oplus \bar{V} \oplus \widehat{V}$ is Maurer–Cartan if and only if it is of the form $\xi + x + 0$ for some $\xi \in \mathrm{MC}(V)$ and $x \in V^0$. Indeed, write an arbitrary degree 1 element as $\xi + x + \eta$ with $\xi \in V$, $x \in \bar{V}$ and $\eta \in \widehat{V}$. Since the brackets are nonzero only on the first component, the Maurer–Cartan equation for $\xi + x + \eta$ reduces to

$$d(\xi) + d(x) + d(\eta) + \sum_{i \geq 2} \frac{1}{i!}[\xi, \ldots, \xi]_i = 0, \tag{4.3.3}$$

Since $d(x) = 0$ by definition, x is arbitrary and we require $d(\eta) = 0 \in \bar{V}$ and $\xi \in \mathrm{MC}(V)$. By abuse of notation, we denote also by ξ its representing cdga map $(SU, d) \to A$, and by x its equivalent degree 0 linear map $\bar{U} \to A$. Hence x and ξ together determine a cdga map $H_{\xi,x}\colon C(SU) \to A$ that is a left homotopy between ξ and $x * \xi \coloneqq H_{\xi,x} \circ e^\theta \circ i$, by

$$H_{\xi,x}(u) = \xi(u),\ H_{\xi,x}(\bar{u}) = x(\bar{u}),\ H_{\xi,x}(\widehat{u}) = 0.$$

We recall the following terminology.

Definition 4.3.3. Let A be an algebra.

(1) A *degree n derivation* of A is a linear map $d\colon A \to A$ of degree n such that

$$d(xy) = d(x)y + (-1)^{n|x|}xd(y)$$

for all homogeneous $x, y \in A$. A derivation with odd (even) degree is called an *odd* (*even*) derivation.

(2) Let $f\colon A \to B$ be a dga map. A linear map $g\colon A \to B$ is an *f-derivation* if

$$g(xy) = g(x)f(y) + f(x)g(y),\ \text{for all } x, y \in A.$$

If $A = B$ and $f = \mathrm{id}$, then an f-derivation is a degree 0 derivation of A.

Note that the square of a constant odd derivation is 0.

LEMMA 4.3.4. *Let $f\colon A \to B$ be a dga map.*

(1) *If $g\colon A \to A$ is a derivation, then fg is an f-derivation.*
(2) *If $g\colon B \to B$ is a derivation, then gf is an f-derivation.*

Our next result gives a compact formula for left homotopy of Maurer–Cartan elements. In the next section, we will show that the formula specialises to gauge equivalence in the case where V is a dg(l)a.

THEOREM 4.3.5.

(1) *Let V be a complete L_∞-algebra. Then two Maurer–Cartan elements $\xi, \eta \in V$ are left homotopic if and only if their representing cdga maps $\xi, \eta\colon S\Sigma^{-1}V^* \to k$ satisfy*
$$\eta = \xi \circ e^{[\tilde{x},d]},$$
where $\tilde{x}$ is the constant degree -1 derivation of $S\Sigma^{-1}V^*$ induced by the left homotopy.
(2) *Let V be a complete A_∞-algebra. Then two Maurer–Cartan elements $\xi, \eta \in V$ are left homotopic if and only if their representing dga maps $\xi, \eta\colon T\Sigma^{-1}V^* \to k$ satisfy*
$$\eta = \xi \circ e^{[\tilde{x},d]},$$
where $\tilde{x}$ is the constant degree -1 derivation of $T\Sigma^{-1}V^*$ induced by the left homotopy.

PROOF. Consider first the L_∞-case. We lift the homotopy $H_{\xi,x}$ between ξ and $x * \xi$ to SU in the following sense: Let f be the cdga map $H_{\mathrm{id},x} \circ e^{\theta} \circ i\colon SU \to SU$. Then $x * \xi = \xi \circ f$ and the identity map of SU is left homotopic to f via $H_{\mathrm{id},x}\colon C(SU) \to SU$, defined by
$$H_{\mathrm{id},x}(u) = u,\ H_{\mathrm{id},x}(\bar{u}) = x(\bar{u}),\ H_{\mathrm{id},x}(\hat{u}) = 0.$$

We now show that $f = e^{[\tilde{x},d]}$, where $\tilde{x}$ is the constant derivation of SU corresponding to x. First convert the homotopy $H_{\mathrm{id},x}$ into a Sullivan homotopy between the identity morphism of SU and f. Consider the map
$$e^{z\theta} + se^{z\theta}dz\colon C(SU) \to C(SU)[z, dz], \tag{4.3.4}$$

which is well-defined as any element $u+\bar{u}+\tilde{u}\in C(SU)$ satisfies $\theta^N(u+\bar{u}+\tilde{u})=0$ for sufficiently large N (see Section 4.3.1), so $e^{z\theta}$ is indeed a polynomial in z. By [BL05, Theorem 3.4], equation (4.3.4) defines a Sullivan homotopy between the identity morphism and the automorphism $e^\theta = e^{[s,D]}$ of $C(SU)$. Then defining $F, G\colon SU \to SU[z]$ to be the compositions

$$F = H_{\mathrm{id},x}\circ e^{z\theta}\circ i,\;\; G = H_{\mathrm{id},x}\circ se^{z\theta}\circ i,$$

we obtain that $F+Gdz\colon SU\to SU[z,dz]$ is a Sullivan homotopy from id to f. Since the constant term of F is always the identity on SU, the map F is formally invertible and the integral formula from [BL05] gives

$$f = \exp\left[\int_0^1 GF^{-1}\,dz,\, d\right].$$

Finally we show that $G=\tilde{x}F$, from which it follows immediately that the integral converges and evaluates to $\tilde{x}$, concluding the proof of the theorem. Indeed, $H_{\mathrm{id},x}s$ and $\tilde{x}H_{\mathrm{id},x}$ are both $H_{\mathrm{id},x}$-derivations $S(U\oplus\bar{U}\oplus\widehat{U})\to SU$, and they agree on $U\oplus\bar{U}\oplus\widehat{U}$:

$$H_{\mathrm{id},x}s(u) = H_{\mathrm{id},x}(\bar{u}) = x(\bar{u}),\;\; H_{\mathrm{id},x}s(\bar{u}) = H_{\mathrm{id},x}(\hat{u}) = 0,\;\; H_{\mathrm{id},x}s(\hat{u}) = 0,$$

and

$$\tilde{x}H_{\mathrm{id},x}(u) = \tilde{x}(u) = x(\bar{u}),\;\; \tilde{x}H_{\mathrm{id},x}(\bar{u}) = \tilde{x}x(\bar{u}) = 0,\;\; \tilde{x}H_{\mathrm{id},x}(\hat{u}) = 0.$$

Hence $H_{\mathrm{id},x}s = \tilde{x}H_{\mathrm{id},x}$, which gives $G=\tilde{x}F$ as required.

Now suppose ξ is a Maurer–Cartan element in an A_∞-algebra. In the A_∞-case, the integral formula no longer applies due to the lack of graded commutativity. However, we can reduce to the L_∞-case as follows. Since k is commutative, its representing dga map $\xi\colon TU\to k$ factors as $\xi=\xi'\circ p$, where $p\colon TU\to SU$ is the canonical projection and $\xi'\colon SU\to k$ is a cdga map. Similarly there are factorizations $x*\xi = (x*\xi)'\circ p$ and $H = H'\circ p$. Then ξ' and $(x*\xi)'$ are Maurer–Cartan elements in the L_∞-algebra represented by SU, and the cdga map H' defines a left homotopy between them. By definition $(x*\xi)' = x*\xi'$, hence by the L_∞-case,

$$x*\xi = (x*\xi')\circ p = \xi'\circ e^{[\tilde{x},d_{SU}]}\circ p = \xi'\circ p\circ e^{[\tilde{x},d_{TU}]} = \xi\circ e^{[\tilde{x},d_{TU}]}.$$

This proves the A_∞-case. □

We would like to extend Theorem 4.3.5 to Maurer–Cartan elements in L_∞-algebras and A_∞-algebras of the form $V \otimes A$, that are not necessarily complete. However, given two left-homotopic Maurer–Cartan elements $\xi, \eta \in V \otimes A$, it is *not* true that their representing cdga maps $\xi, \eta \colon S\Sigma^{-1}V^* \to A$ satisfy $\eta = \xi \circ e^{[\tilde{x},d]}$ for some degree -1 derivation $\tilde{x}$ of $S\Sigma^{-1}V^*$, as the following counterexample shows.

EXAMPLE 4.3.6. Take (V, d) to be a dg vector space, so that it has a linear differential and a decomposition $V = H(V) \oplus \Sigma B \oplus B$, where $H(V)$ is the homology of V and $B^n = \operatorname{im} d^{n-1} \subseteq V^n$. Take $A = S\Sigma^{-1}V^*$ and let ξ be the identity map on A and η be the endomorphism on A induced by the projection of V onto $H(V)$. Then ξ and η are left homotopic, but η is not an automorphism, so cannot be of the form $\mathrm{id} \circ e^{[\tilde{x},d]}$. Indeed, if $\tilde{x}$ is the constant derivation corresponding to the homotopy, then the exponential $e^{[\tilde{x},d]}$ diverges.

To obtain an analogue of Theorem 4.3.5 for Maurer–Cartan elements in $V \otimes A$ requires introducing a *semi-completed symmetric algebra* and a *semi-completed tensor algebra*: for a pseudocompact vector space V, define

$$S'V = \bigoplus_{i \geq 0} S^i V \quad \text{and} \quad T'V = \bigoplus_{i \geq 0} T^i V.$$

Since V is pseudocompact, $S^i(V)$ and $T^i(V)$ are still assumed to mean completed tensor powers. However, $S'V$ and $T'V$ differ from $\widehat{S}V$ and $\widehat{T}V$ by taking the direct sum of tensor powers instead of the direct product. Thus $S'V$ and $T'V$ are not pseudocompact, but do have some non-discrete topology. They have the following property.

LEMMA 4.3.7. *Let V be a pseudocompact vector space.*

(1) *Let B be a pseudocompact (c)dga. Any continuous linear map $V \to B$ extends uniquely to a continuous cdga map $S'V \to B$ or a continuous dga map $T'V \to B$.*
(2) *Any continuous linear map $V \to S'V$ extends uniquely to a continuous derivation of $S'V$, and any continuous linear map $V \to T'V$ extends uniquely to a continuous derivation of $T'V$.*

PROOF.

(1) Since elements of $S'V$ and $T'V$ are finite sums of tensor powers, it suffices to prove that a linear map $f \colon V = \varprojlim_i V_i \to B$ extends to continuous

maps $V^{\widehat{\otimes} n} \to B$ for all $n \geq 2$. Since f is determined by $V_i \to B$, we define $f^{\otimes n}\colon V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n} \to B$, and take the projective limit to obtain a map on $V^{\widehat{\otimes} n}$.

(2) This follows the same argument as above, but instead we extend f to $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n}$ by $\sum_{j=0}^{n-1} 1^j \otimes f \otimes 1^{n-1-j}$. □

This allows us to give an alternative characterization of Maurer–Cartan elements as continuous (c)dga maps.

Lemma 4.3.8. *Let V be a finite-dimensional complete L_∞-algebra, and let A be a cdga. There is a correspondence*

$$\mathrm{MC}(V \otimes A) \cong \mathrm{Hom}(S'\Sigma^{-1}(V \otimes A)^*, k).$$

Let V be a finite-dimensional complete A_∞-algebra. There is a correspondence

$$\mathrm{MC}(V \otimes A) \cong \mathrm{Hom}(T'\Sigma^{-1}(V \otimes A)^*, k).$$

We recover the usual representing (c)dga maps if V is finite-dimensional and $A = k$.

Proof. We treat the L_∞-case only. First note that the object $S'\Sigma^{-1}(V \otimes A)^*$ makes sense: V is finite-dimensional, so $V \otimes A$ is discrete and its dual is pseudocompact. Recall that $V \otimes A \cong ((V \otimes A)^*)^*$. So there is a correspondence between $(\Sigma(V \otimes A))^0$ and continuous degree 0 linear maps $\Sigma^{-1}(V \otimes A)^* \to k$, which correspond to continuous degree 0 algebra maps $S'\Sigma^{-1}(V \otimes A)^* \to k$ by Lemma 4.3.7.

The Maurer–Cartan condition corresponds to the correct axioms for differentials on $S'\Sigma^{-1}(V \otimes A)^*$ by the same argument as the usual (non-continuous) cdga case in Proposition 3.3.4: Apply Lemma 4.3.7 again to replace the commutative square in the proof of Proposition 3.3.4 by

$$\begin{array}{ccccc} \Sigma^{-1}(V \otimes A)^* & \xrightarrow{d_n} & (\Sigma^{-1}(V \otimes A)^*)^{\otimes n} & \hookrightarrow & S'\Sigma^{-1}(V \otimes A)^* \\ \downarrow & & & & \downarrow \\ k & & \longrightarrow & & k \end{array}$$

The rest of the proof is identical. □

Lemma 4.3.8 allows us to imitate the previous case when $A = k$, replacing V by $V \otimes A$. Let V be a finite-dimensional complete L_∞-algebra, and let A be a cdga.

We proceed as at the start of the section: consider $(V \otimes A) \oplus \overline{V \otimes A} \oplus \widehat{V \otimes A}$ with notation, differential and brackets as in (4.3.2). For $U = \Sigma^{-1}(V \otimes A)^*$, consider

$$C(S'U) := (S'(U \oplus \bar{U} \oplus \widehat{U}), D),$$

where $\bar{U} \cong \Sigma U$ and $\widehat{U} \cong U$, and D are defined as for $C(SU)$ in Section 4.3.1.

The same computation as (4.3.3) shows that an element in $(V \otimes A) \oplus \overline{V \otimes A} \oplus \widehat{V \otimes A}$ satisfies the Maurer–Cartan equation if and only if it is of the form $\xi + x + 0$ for some $\xi \in \mathrm{MC}(V \otimes A)$ and $x \in (V \otimes A)^0$. Hence, by Lemma 4.3.8, x and ξ together determine a continuous cdga map $H\colon C(S'U) \to k$.

Theorem 4.3.9.

(1) *Let V be a finite-dimensional complete L_∞-algebra and A be a cdga. Two Maurer–Cartan elements $\xi, \eta \in V \otimes A$ are left homotopic if and only if their representing continuous cdga maps $\xi', \eta'\colon S'\Sigma^{-1}(V \otimes A)^* \to k$ satisfy*

$$\eta' = \xi' \circ e^{[\tilde{x}', d']},$$

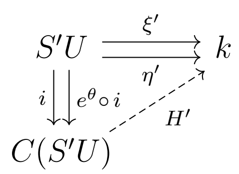
where $\tilde{x}'$ is the constant degree -1 derivation of $S'\Sigma^{-1}(V \otimes A)^*$ induced by the left homotopy.

(2) *Let V be a finite-dimensional complete A_∞-algebra and A be a cdga. Two Maurer–Cartan elements $\xi, \eta \in V \otimes A$ are left homotopic if and only if their representing continuous dga maps $\xi', \eta'\colon T'\Sigma^{-1}(V \otimes A)^* \to k$ satisfy*

$$\eta' = \xi' \circ e^{[\tilde{x}', d']},$$

where $\tilde{x}'$ is the constant degree -1 derivation of $T'\Sigma^{-1}(V \otimes A)^*$ induced by the left homotopy.

Proof. We prove the L_∞-case; the A_∞-case can be reduced to the L_∞-case as before. While the object $C(S'U)$ defined above is not a cylinder object, we can treat it as if it were one. As before, we can define an automorphism $e^\theta = \sum_{n=0}^{\infty} \theta^n/n!$ of $C(S'U)$; note that the Sullivan condition still holds on the differential D of $S'U$, so the series still converges. Then by Lemma 4.3.7 and Lemma 4.3.8, ξ and η are left homotopic if and only if there is a cdga map H' such that the diagram commutes:



The rest of the proof is the same as Theorem 4.3.5, replacing $C(SU)$ with $C(S'U)$ everywhere. The Sullivan condition still holds on the differential d' of $S'U$, so the exponential $e^{[\tilde{x}',d']}$ converges as before. □

4.4. Left homotopy and gauge equivalence

In this section, Theorem 4.3.5 and Theorem 4.3.9 are used to obtain combinatorial formulae for the Maurer–Cartan element $x * \xi = H_{\xi,x} \circ e^{\theta} \circ i$, in terms of rooted trees. The formulae will show that Theorem 4.3.5 and Theorem 4.3.9 specialize to gauge equivalence in the case of a dgla or dga. We will use this to deduce the following theorem at the end of the section.

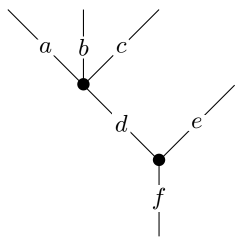
THEOREM 4.4.1. *Let V be a complete dgla and A be a cdga. For any two Maurer–Cartan elements ξ and η in $V \otimes A$, the following are equivalent:*

(1) *ξ and η are gauge equivalent;*
(2) *ξ and η are left homotopic;*
(3) *ξ and η are Sullivan homotopic.*

A proof that (1) and (3) coincide already appears in the literature; see for example [SS] and [CL10, Theorem 4.4]. The originality of Theorem 4.4.1 lies in directly establishing the equivalence of (1) and (2). The proof that (2) and (3) coincide is purely model category theoretic, so this theorem also provides a new proof that gauge equivalence and Sullivan homotopy coincide.

The terminology and conventions that we will use for rooted trees mainly follow those of [GK94, Section 1.1], which we recall now for convenience. A *tree* is a nonempty connected oriented graph without loops, such that each vertex has at least one incoming and one outgoing edge. Some edges can be bound by a vertex at one end only; such edges are called *external* and other edges are called *internal*. A *rooted tree* has a distinguished external edge called the *root*. All the other external edges are called *leaves*; the edges of a rooted tree can therefore be partitioned into the root, the leaves and the internal edges. There is also a single *degenerate* tree with no vertices, and a single leaf.

EXAMPLE 4.4.2. The following tree has two vertices, root f, four leaves $\{a, b, c, e\}$ and internal edge d.



A tree with exactly one vertex and n leaves is called an *n-star*. Given two trees T_1 and T_2, the *composition of T_1 and T_2* along an edge $j \in T_2$ is the tree obtained by identifying the root of T_1 with the jth leaf of T_2. Additionally, given a rooted tree T and a natural number k, we say that the *maximal height k sub-tree of T* is the rooted sub-tree of T consisting of all vertices with a path to the root with length at most k, together with their internal edges and leaves.

THEOREM 4.4.3. *Let V be a complete L_∞-algebra, and let $x \in V^0$, $\xi \in \mathrm{MC}(V)$. Then*

$$x * \xi = \sum_T \frac{(-1)^n r}{n! j_1! \dots j_n!} T(x, \xi),$$

where the sum is taken over all rooted trees T such that every vertex has at least one leaf, and for each rooted tree T,

1. *n is the number of vertices of T;*
2. *r is the number of orderings of the vertices of T such that each vertex is greater than its parent;*
3. *$T(x, \xi)$ is the unique word associated to T given by labelling exactly one leaf on each vertex by x and all remaining leaves by ξ, and associating to each degree i vertex with inputs $\eta_1, \dots, \eta_{i-1}, x$ the operation $[\eta_1, \dots, \eta_{i-1}, x]$;*
4. *$j_1, \dots, j_n$ are the numbers of ξ attached to each of the n vertices.*

PROOF. As before, we write $U = \Sigma^{-1} V^*$. Using that $\tilde{x}^2 = 0$ (as the square of a constant odd derivation is 0), the same calculation as for the series e^θ in equation (4.3.1) gives

$$e^{[\tilde{x},d]}(u) = u + \sum_{n=1}^{\infty} \frac{(\tilde{x}d)^n(u)}{n!}$$

for $u \in U$. From Theorem 4.3.5 and Theorem 4.3.9, the Maurer–Cartan element $x * \xi$ is represented by the cdga map $\xi \circ e^{[\tilde{x},d]} : SU \to k$, so the restriction of $\xi \circ e^{[\tilde{x},d]}$ to U is $\mathrm{ev}_{x*\xi}$. Applying $\xi \circ e^{[\tilde{x},d]}$ to an element $u \colon \Sigma V \to k$ in U is equivalent to forming a tree by successive compositions. Since $\tilde{x}$ and d are derivations, each

$(\tilde{x}d)^n(u)$ is a sum of words determined by sequences d_{i_1}, d_{i_2}, $\dots$, d_{i_n}, for any $i_1, \dots, i_n \geq 1$. Hence at each step:

(1) Applying d_i to $u \in U$ gives the composition $\frac{1}{i!} u \circ \ell_i$. Applying $\mathrm{id}^{\otimes j-1} \otimes d_i \otimes \mathrm{id}^{\otimes k-j}$ to an element of $U^{\otimes k}$ therefore corresponds to composition with an i-star along j.

(2) Applying $\tilde{x}$ to $u \in U$ is the evaluation $\mathrm{ev}_x(u)$. Applying $\mathrm{id}^{\otimes j-1} \otimes \tilde{x} \otimes \mathrm{id}^{\otimes k-j}$ to an element of $U^{\otimes k}$ therefore corresponds labelling the jth leaf with x.

(3) Applying ξ to $u \in U$ is the evaluation $\mathrm{ev}_\xi(u)$. Since ξ extends to a cgda map, this corresponds to labelling all remaining leaves with ξ.

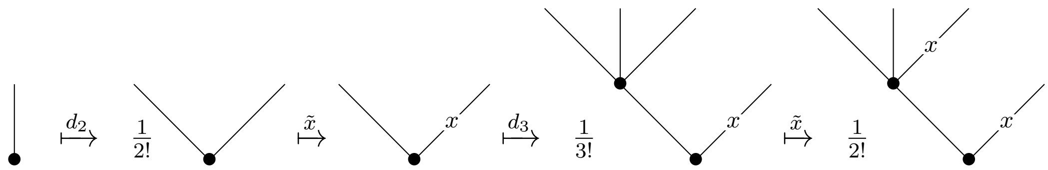
Every sequence d_{i_1}, d_{i_2}, $\dots$, d_{i_n} gives words of the form $(-1)^n T(x, \xi)/i_1! \dots i_n!$. Indeed, regard ξ and x as elements in ΣV, so that ξ and x have degrees 0 and -1 respectively. Then by graded symmetry of the ℓ_i, there must be exactly one x on each vertex, and every term can be written as $\ell_i(\eta_1, \dots, \eta_{i-1}, x)$, which equals $[\eta_1, \dots, \eta_{i-1}, x]$ by our grading convention. Finally, each $\tilde{x}d$ introduces a sign -1, as both ℓ_i and x both have odd degree.

To determine the coefficient, it remains to count how many ways compositions give rise to the same word. By graded commutativity, we can form the trees such that each composition or labelling by x always fills the last unlabelled leaf on each vertex. With this restriction, the number of ways a tree can be built is r, the number of monotone orderings of its vertices. □

Example 4.4.4. We illustrate the combinatorics of the proof in a simple example. Consider the sequence $\tilde{x}d_3\tilde{x}d_2$:

$$U \xrightarrow{d_2 = \frac{1}{2!}\ell_2^*} U^{\otimes 2} \xrightarrow{\tilde{x}} U \xrightarrow{d_3} U^{\otimes 3} \xrightarrow{\tilde{x}} U^{\otimes 2}$$

This corresponds to building a rooted tree via the following sequence (we omit drawing the root for simplicity).



Note that every time $\tilde{x}$ follows a d_i, there are i choices for labelling x. In this case the number of monotone orderings of the vertices is simply $r = 1$ (higher values of r are possible for higher d_i, by composing trees along different edges attached to the same vertex), and we have $n = 2$, $j_1 = 2 - 1 = 1$, $j_2 = 3 - 1 = 2$. The resulting term in $x * \xi$ is $\frac{1}{2!2!}[[\xi, \xi, x], x]$.

When V is a dgla, the above formula only allows rooted trees with vertices of valence 1 or 2, and the coefficients r, j_1, ..., j_n equal 1 for every tree. This recovers the formula (4.1.1) for gauge equivalence in dglas, and recovers the formula of [Get09, Proposition 5.9] in the case of L_∞-algebras.

Similarly we can use Theorem 4.3.5 and Theorem 4.3.9 to obtain the following analogous formula for the A_∞-case.

THEOREM 4.4.5. *Let A be a complete A_∞-algebra, and let $x \in A^0$, $\xi \in \mathrm{MC}(A)$. Then*

$$x * \xi = \sum_T \sum_\lambda \frac{(-1)^n}{n!} T_\lambda(x, \xi),$$

where the sum is taken over all planar rooted trees T, and for each rooted tree T,

(1) *n is the number of vertices of T;*
(2) *λ ranges over labellings of T that label n leaves by x and all remaining leaves by ξ, such that for any $1 \leq k \leq n$, the maximal height k sub-tree of T has k leaves labelled by x;*
(3) *$T_\lambda(x, \xi)$ is the word given by the labelling λ and associating to each degree i vertex the operation $m_i \colon T^i\Sigma A \to \Sigma A$.*

PROOF. The calculation is similar to the L_∞-case in Theorem 4.4.3, except the lack of graded commutativity means that each m_i can take more than one x. Each sequence d_{i_1}, d_{i_2}, ..., d_{i_n} gives words of the form $(-1)^n T_\lambda(x, \xi)$. □

PROOF OF THEOREM 4.4.1. First note that the equivalence of (2) and (3) is immediate by completeness of V. Also, if V is finite-dimensional, then the equivalence of (1) and (2) is immediate by Theorem 4.3.9 and Theorem 4.4.3. Finally in the infinite dimensional case, by completeness of V we have $V = \varprojlim V_i$ where V_i are all finite-dimensional complete dglas. Hence (1) and (2) are equivalent for V, as they are equivalent for each V_i. □

REMARK 4.4.6. In the case where $V \otimes A$ is complete (in particular, when $A = k$), we can simplify the above proof, as the result is just a direct consequence of Theorem 4.3.5 and Theorem 4.4.3.

4.5. A strong homotopy Poincaré lemma

The Poincaré lemma states that on a contractible manifold, every closed differential form of positive degree is exact. The following non-abelian analogue to the Poincaré lemma is proved in [Vor12, Theorem 3.1].

THEOREM 4.5.1 (Non-abelian Poincaré lemma). *Let M be a contractible manifold and let $\mathfrak{g}$ be a dgla. Let ξ be a $\mathfrak{g}$-valued differential form on M such that ξ is a Maurer–Cartan element in $\mathfrak{g} \otimes \Omega(M)$. Then ξ is gauge equivalent to a constant.*

EXAMPLE 4.5.2. For the simplest example of this theorem, suppose $\mathfrak{g} = \mathbb{R}$; this is trivially a dgla by setting $d_{\mathfrak{g}} = 0$, $[-,-]_{\mathfrak{g}} = 0$, and $\mathbb{R}$ to be concentrated in degree 0. An $\mathbb{R}$-valued differential form of degree 1 is just a form $\omega \in \Omega^1(M)$. Since the brackets vanish, ω being Maurer–Cartan just means

$$d\omega = 0, \ \omega \in \Omega^1(M),$$

and ω being gauge equivalent (4.1.1) to a constant means there is some Maurer–Cartan element $C = C \otimes 1 \in \mathbb{R} \otimes \Omega(M)$ such that

$$\omega = e^\sigma \cdot C = C - d\sigma \text{ for some } \sigma \in \Omega^0(M).$$

However, $C \otimes 1$ has total degree 1, forcing $C \in \mathfrak{g}^1 = 0$. Hence the theorem says a closed 1-form is exact, recovering the usual Poincaré lemma for 1-forms.

It was suggested in [Vor12] that Theorem 4.5.1 may be extended to the setting of L_∞-algebras. Here, we prove such a statement as an application of the results from the previous sections, by interpreting the left homotopy defined in Definition 4.3.1 as a gauge equivalence in the case of L_∞-algebras, where there is no independent notion of gauge equivalence.

We note also that [Vor12] considers different notions of homotopy and gauge equivalence, which hold for arbitrary odd elements in dg Lie superalgebras (admitting a gauge group, in the case of gauge equivalence), not just Maurer–Cartan elements. For Maurer–Cartan elements of complete dglas, the notion of homotopy considered in [Vor12] coincides with Sullivan homotopy by [Vor12, Remark 5.2], and gauge equivalence also coincides by a generalization of the Schlessinger–Stasheff theorem [Vor12, Theorem 5.2].

THEOREM 4.5.3 (Strong homotopy Poincaré lemma). *Let M be a contractible manifold and let $\mathfrak{g}$ be a complete L_∞-algebra. If ξ is a $\mathfrak{g}$-valued differential form on M that is Maurer–Cartan, then ξ is gauge equivalent to a constant.*

PROOF. If $\mathfrak{g}$ is a complete dgla, then by Theorem 4.4.1, two Maurer–Cartan elements in $\mathfrak{g}\otimes\Omega(M)$ are gauge equivalent if and only if their representing cdga maps are left homotopic. More generally, the explicit formulas obtained in Theorem 4.4.3, coinciding with those in [Get09], allow us to interpret left homotopy as a gauge equivalence for Maurer–Cartan elements in L_∞-algebras.

Now, since M is contractible, $\Omega(M)$ is weakly equivalent to $\mathbb{R}$. By completeness of $\mathfrak{g}$, homotopy classes of maps $S\Sigma^{-1}\mathfrak{g} \to \Omega(M)$ correspond to homotopy classes of maps $S\Sigma^{-1}\mathfrak{g} \to \mathbb{R}$. Hence $\mathscr{MC}(\mathfrak{g}, \Omega(M)) \cong \mathscr{MC}(\mathfrak{g})$. □

CHAPTER 5

Koszul duality for compactly generated derived categories of second kind

In this chapter, we use the extended bar-cobar adjunction to associate "extended Koszul dual" dg coalgebras to dg algebras. These extended Koszul duals will be given by an "extended bar construction", and the resulting coalgebras are generally much larger than the Koszul duals given by the usual bar construction; they are also not conilpotent in general. Our goal is to prove that there is a Quillen equivalence between model categories of dg modules over a dg algebra and comodules over its extended Koszul dual dg coalgebra.

Of particular interest is the exotic model structure on dg modules, where the weak equivalences for dg modules are now also of second kind. Model categories of second kind arise naturally in the context of Koszul duality, where in order to remove boundedness conditions, one is led to consider a notion of weak equivalence that is more subtle than quasi-isomorphism [Hin01, LH03]. On the comodule side, derived categories of second kind also have the advantage of being more well-behaved than the usual derived category. For example, it is well known that the derived category $D(A)$ of dg A-modules for a dg algebra A is compactly generated by A itself, and the compact objects are precisely perfect complexes. On the other hand, it is unclear if the derived category $D(C)$ of dg C-comodules for a dg coalgebra C is compactly generated; however, its derived category of second kind is indeed compactly generated.

The main idea in the proof of the model structure is to identify the weak equivalences using a class of dg modules, called "twisted dg modules", which are thought of as prototype cofibrant objects. These twisted modules have appeared in the study of derived categories of coherent sheaves on complex analytic manifolds and infinity local systems on topological spaces [CHL21].

5.1. Extended bar construction

Given an algebra A, its *pseudocompact completion* $\check{A}$ is the projective limit of the inverse system of quotients by cofinite-dimensional ideals of A. Pseudocompact completion defines a functor from $\mathsf{Alg} \to \mathsf{pcAlg}$ that is left adjoint to the functor $\mathsf{pcAlg} \to \mathsf{Alg}$ forgetting the topology.

Definition 5.1.1. Let V be a pseudocompact graded vector space. If V is finite-dimensional, its *pseudocompact tensor algebra* $\check{T}V$ is the pseudocompact completion of the tensor algebra TV. For a general pseudocompact vector space $V = \varprojlim_i V_i$, its *pseudocompact tensor algebra* is

$$\check{T}V \coloneqq \varprojlim_i \check{T}V_i.$$

Proposition 5.1.2. *Let V be a pseudocompact graded vector space.*

(1) *The pseudocompact tensor algebra $\check{T}V$ is the free pseudocompact algebra on V, that is, for any pseudocompact algebra A there is a bijection*

$$\mathrm{Hom}_{\mathsf{pcAlg}}(\check{T}V, A) \cong \mathrm{Hom}(V, A).$$

(2) *For any pseudocompact $\check{T}V$-$\check{T}V$-bimodule M there is a bijection*

$$\mathrm{Der}(\check{T}V, M) \cong \mathrm{Hom}(V, M).$$

Proof.

(1) If V is finite-dimensional, then V is discrete and

$$\mathrm{Hom}(V, A) \cong \mathrm{Hom}_{\mathsf{Alg}}(TV, A),$$

which equals $\mathrm{Hom}_{\mathsf{pcAlg}}(\check{T}V, A)$ as pseudocompact completion is left adjoint to the forgetful functor. More generally, writing $V = \varprojlim_i V_i$ and $A = \varprojlim_j A_j$ with V_i and A_j finite-dimensional, we have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{pcAlg}}(\check{T}V, A) &\cong \varprojlim_j \mathrm{Hom}_{\mathsf{pcAlg}}(\check{T}V, A_j)\\ &\cong \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathsf{pcAlg}}(\check{T}V_i, A_j)\\ &\cong \varprojlim_j \varinjlim_i \mathrm{Hom}(V_i, A_j) \cong \mathrm{Hom}(V, A).\end{aligned}$$

Here, the second bijection holds as finite-dimensional algebras are cocompact in pcAlg, that is, for any finite-dimensional algebra A, the functor

$\mathrm{Hom}_{\mathsf{pcAlg}}(-, A)$ takes projective limits to inductive colimits. This is formally dual to the statement that finite-dimensional coalgebras are compact objects; see [GG99, Lemma 1.9] for a proof.

(2) Recall the following construction, which allows us to turn questions about derivations into question about algebra homomorphisms. Given a graded pseudocompact algebra A and an A-A-bimodule M consider the pseudocompact algebra $A \oplus M$ with multiplication $(a, m) \cdot (b, n) = (ab, an + mb)$, and let $p \colon A \oplus M \to A$ be the natural projection. Then there is a bijection $\mathrm{Der}(A, M) \cong \{f \in \mathrm{Hom}_{\mathsf{pcAlg}}(A, A \oplus M) : p \circ f = 1_A\}$. Setting $A = \check{T}V$ and using part (1), we have

$$\begin{aligned}\mathrm{Der}(\check{T}V, M) &\cong \{f \in \mathrm{Hom}(V, \check{T}V \oplus M) : p \circ f = 1_{\check{T}V}|_V\} \\ &\cong \mathrm{Hom}(V, M).\end{aligned}$$

□

Remark 5.1.3. The pseudocompact algebra $\check{T}V$ is the k-linear dual to the Sweedler cofree coalgebra on the discrete vector space V^*, [Swe69, Section 6.4].

Proposition 5.1.4. *For any pseudocompact vector space V, there is a bimodule resolution of $\check{T}V$ given by*

$$0 \longrightarrow \check{T}V \otimes V \otimes \check{T}V \xrightarrow{d} \check{T}V \otimes \check{T}V \xrightarrow{m} \check{T}V \longrightarrow 0$$

where m is multiplication and $d(1 \otimes v \otimes 1) = v \otimes 1 - 1 \otimes v$.

Proof. We use the following well-known fact for algebras that also holds in the pseudocompact setting. Let A be a graded pseudocompact algebra with multiplication $\mu \colon A \otimes A \to A$. Then $\Omega(A) = \ker \mu$ is an A-A-bimodule and the map $\delta \colon A \to \Omega(A)$ given by $\delta(a) = a \otimes 1 - 1 \otimes a$ is a derivation. For any derivation $d \colon A \to M$ taking values in an A-A-bimodule M, there is a unique bimodule homomorphism $f \colon \Omega(A) \to M$ such that $d = f \circ \delta$; hence

$$\mathrm{Der}(A, M) \cong \mathrm{Hom}_{A\text{-}A}(\Omega(A), M).$$

Now by Proposition 5.1.2,

$$\mathrm{Der}(\check{T}V, M) \cong \mathrm{Hom}(V, M) \cong \mathrm{Hom}_{\check{T}V\text{-}\check{T}V}(\check{T}V \otimes V \otimes \check{T}V, M),$$

so $\Omega(\check{T}V) \cong \check{T}V \otimes V \otimes \check{T}V$ as required. □

All our dg algebras are augmented, except in Section 5.3. The augmentation ideal of a dg algebra A is denoted by $\bar{A}$.

Definition 5.1.5. We define a pair of functors

$$\Omega\colon (\mathsf{pcDGA}^*)^{\mathsf{op}} \rightleftarrows \mathsf{DGA}^* : \check{B}$$

as follows. The *cobar construction* associates to a pseudocompact dg algebra C the dg algebra

$$\Omega C := T\Sigma^{-1}\bar{C}^*$$

with differential defined in the usual way.

The *extended bar construction* associates to a dg algebra A the pseudocompact dg algebra

$$\check{B}A := \check{T}\Sigma^{-1}\bar{A}^*.$$

We define the differential on $\check{B}A$ as follows: Let $d_1\colon \Sigma^{-1}\bar{A}^* \to \Sigma^{-1}\bar{A}^*$ and $d_2\colon \Sigma^{-1}\bar{A}^* \to \Sigma^{-1}\bar{A}^*\widehat{\otimes}\Sigma^{-1}\bar{A}^*$ be induced by dualising the differential and multiplication on A respectively. For a pseudocompact vector space V, consider the *semi-completed tensor algebra* $T'(V) = \bigoplus_{n\geq 1} V^{\widehat{\otimes} n}$, which has a topology that is neither pseudocompact nor discrete, and has the property $\mathrm{Hom}_{\mathsf{Alg}}(T'(V), B) \cong \mathrm{Hom}(V, B)$ for any pseudocompact algebra B, by Lemma 4.3.7. Then by Proposition 5.1.2(1), the identity on $\check{T}\Sigma^{-1}\bar{A}^*$ induces a map $i\colon T'(\Sigma^{-1}\bar{A}^*) \to \check{T}(\Sigma^{-1}\bar{A}^*)$, and we define the differential to be

$$i \circ (d_1 + d_2)\colon \Sigma^{-1}\bar{A}^* \to T'(\Sigma^{-1}\bar{A}^*) \to \check{T}(\Sigma^{-1}\bar{A}^*).$$

5.1.1. The Maurer–Cartan functor and representability. Let A be a dg algebra (possibly discrete, pseudocompact or otherwise). A *Maurer–Cartan element* in A is an element $x \in A$ of degree 1 such that $dx + x^2 = 0$. The set of all Maurer–Cartan elements in A is denoted by $\mathrm{MC}(A)$. For any dg algebra A and any pseudocompact dg algebra C, define $\mathrm{MC}(A, C) := \mathrm{MC}(A \otimes C)$; this is functorial in both arguments.

Proposition 5.1.6. *Let A be an augmented dg algebra and C be an augmented pseudocompact dg algebra. There are natural bijections*

$$\mathrm{Hom}_{\mathsf{DGA}^*}(\Omega C, A) \cong \mathrm{MC}(\bar{A}, \bar{C}) \cong \mathrm{Hom}_{\mathsf{pcDGA}^*}(\check{B}A, C).$$

In particular, Ω is a left adjoint functor to $\check{B}$.

Proof. Forgetting the differential, any map of augmented pseudocompact algebras $f\colon \check{B}A \to C$ is equivalent to a linear map $\Sigma^{-1}\bar{A}^* \to \bar{C}$ by Proposition 5.1.2, which is equivalently a degree 1 element $x \in \bar{A} \otimes \bar{C}$. Note that the map lands in $\bar{C}$ as it is augmented.

The condition that f commutes with differentials is then equivalent to condition that x satisfies the Maurer–Cartan equation; this can be proven just like the corresponding statement for the non-extended bar construction, see the proof of Proposition 3.3.14 and Proposition 3.3.4. The other bijection is proved similarly. □

REMARK 5.1.7. An adjoint pair of functors $(\Omega, \mathrm{B}^{\mathrm{ext}})$ between DGA^* and pcDGA^* was defined in [AJ, Section 5.3] in a different way; it was also proved that that these functors represent the Maurer–Cartan sets (called "twisting cochains" in [AJ]) as in Proposition 5.1.6. It follows that these functors are (isomorphic to) the functors Ω and $\breve{B}$ defined above.

5.2. Extended Koszul duality for modules

We begin by recalling the notion of Maurer–Cartan twistings of dg algebras and dg modules, and recalling the standard formulation of Koszul duality for modules, which we will later generalize.

DEFINITION 5.2.1. Let (A, d_A) be a dg algebra and $x \in \mathrm{MC}(A)$.

(1) The *twisted algebra of A by x*, denoted by $A^x = (A, d^x)$, is the dg algebra with the same underlying algebra as A and differential $d^x(a) = d_A(a) + [x, a]$.
(2) Let (M, d_M) be a *left* dg A-module. The *twisted module of M by x*, denoted by $M^{[x]} = (M, d^{[x]})$, is the left dg A^x-module with the same underlying module structure as M and differential $d^{[x]}(m) = d(m) + xm$.

Furthermore, if A and B are dg algebras and M is a dg A-B-bimodule, then for any $x \in \mathrm{MC}(A)$ the twisted module of M by x is a dg A^x-B-bimodule, that is, the right B-module action remains compatible with the new differential.

DEFINITION 5.2.2. A *twisted A-module* is a dg A-module that is free as an A-module after forgetting the differential, that is, it is isomorphic as an A-module to $V \otimes A$ for some graded vector space V. A *finitely generated twisted A-module* is a twisted A-module $V \otimes A$ with V finite-dimensional.

Given any graded vector space V, the A-module $V \otimes A$ equipped with the differential $1 \otimes d_A$ is a twisted A-module. More generally, by considering $V \otimes A$ as a

$(\mathrm{End}(V)\otimes A)$-$A$-bimodule, every twisted A-module is of the form $(V\otimes A, 1\otimes d_A)^{[x]}$ for some $x\in \mathrm{MC}(\mathrm{End}\, V\otimes A)$, as noted in [CHL21, Remark 3.2].

Definition 5.2.3. Let A be an augmented dg algebra, and let $\check{B}A$ be its extended bar construction. Let $\xi\in\mathrm{MC}(A\otimes\check{B}A)$ be the canonical Maurer–Cartan element corresponding to the counit $\Omega\check{B}A\to A$ of the adjunction $\Omega\dashv\check{B}$. Define a pair of functors

$$G\colon (\mathsf{pcDGMod}\text{-}\check{B}A)^{\mathsf{op}} \rightleftarrows \mathsf{DGMod}\text{-}A : F$$

as follows. The functor F associates to a dg A-module M the pseudocompact dg $\check{B}A$-module

$$FM := (M^*\otimes\check{B}A)^{[\xi]}$$

and the functor G associates to a pseudocompact dg $\check{B}A$-module N the dg A-module

$$GN := (N^*\otimes A)^{[\xi]}.$$

The functors F and G are well-defined as FM is a dg $(A\otimes\check{B}A)^{\xi}$-$\check{B}A$-bimodule and GN is a dg $(\check{B}A\otimes A)^{\xi}$-$A$-bimodule; the left $(A\otimes\check{B}A)^{\xi}$-module structure on FM is disregarded as similarly with GN. It is a standard fact that G is left adjoint to F; more generally this is true replacing $\check{B}A$ with any pseudocompact dg algebra C and ξ with any Maurer–Cartan element in $A\otimes C$, see for example [Pos11, Section 6.2].

Remark 5.2.4. In the standard formulation of Koszul duality, the functors are defined as follows: the bar construction of a dg algebra A is instead defined to be $BA=\widehat{T}\Sigma^{-1}\bar{A}^*$, a *local* or *pronilpotent* pseudocompact dg algebra (or dually, a conilpotent dg coalgebra). Given a dg A-module M, the corresponding BA-module is defined as $(M^*\otimes BA)^{[\xi]}$ where $\xi\in\mathrm{MC}(A\otimes BA)$ is the canonical Maurer–Cartan element corresponding to the counit $\Omega BA\to A$ of the Koszul duality adjunction for algebras. Conversely, given a BA-module N, the corresponding A-module is defined as $(N^*\otimes A)^{[\xi]}$.

5.2.1. Model category structure on DGMod-A. We now define model category structures on DGMod-A and $\mathsf{pcDGMod}$-$\check{B}A$ making the adjunction $G\dashv F$ a Quillen pair. In [Pos11] Positselski constructs a model category structure of the "second kind" on the category of dg comodules over an arbitrary (not necessarily conilpotent) dg coalgebra; this will be the model category structure on $\mathsf{pcDGMod}\text{-}\check{B}A$. We begin by recalling this result.

Definition 5.2.5. Let C be a dg coalgebra. A dg C-comodule is *coacyclic* if it is in the minimal triangulated subcategory of the homotopy category of dg C-comodules containing the total C-comodules of exact triples of dg C-comodules and closed under infinite direct sums.

Theorem 5.2.6. [Pos11, Theorem 8.2] *Let C be a dg coalgebra. There exists a model category structure on the category of dg C-comodules, where*

(1) *a morphism $f\colon M \to N$ is a* weak equivalence *if its cone is a coacyclic dg C-comodule;*
(2) *a morphism is a* cofibration *if it is injective;*
(3) *a morphism is a* fibration *if it is surjective with a fibrant kernel.*

Furthermore, this model category structure is cofibrantly generated, where generating cofibrations are injective maps between finite-dimensional comodules. Its homotopy category is a compactly generated triangulated category, with compact generators being finite-dimensional dg comodules; see [Pos11, Section 5.5]. This allows us to dualize Theorem 5.2.6 to get a model category structure as follows:

Theorem 5.2.7. *Let A be a pseudocompact dg algebra. There exists a model category structure on the category of pseudocompact dg A-modules, where*

(1) *a morphism $f\colon M \to N$ is a* weak equivalence *if it induces a quasi-isomorphism*
$$\mathrm{Hom}_A(M, V) \to \mathrm{Hom}_A(N, V)$$
for any finite-dimensional dg $\check{B}A$-module V;
(2) *a morphism is a* fibration *if it is surjective;*

Theorem 5.2.8. *Let A be an augmented dg algebra. There is a cofibrantly generated model category structure on* $\mathsf{DGMod}\text{-}A$*, where*

(1) *a morphism $f\colon M \to N$ is a* weak equivalence *if it induces a quasi-isomorphism*
$$\mathrm{Hom}_A((V \otimes A)^{[x]}, M) \to \mathrm{Hom}_A((V \otimes A)^{[x]}, N)$$
for any finitely generated twisted A-module $(V \otimes A)^{[x]}$;

(2) *a morphism is a* fibration *if it is surjective;*

(3) *a morphism is a* cofibration *if it has the left lifting property with respect to acyclic fibrations.*

With this model structure, the adjunction $G \dashv F$ *is a Quillen pair.*

To prove Theorem 5.2.8, we will apply the following version of the transfer principle, which appears in [BM03, Sections 2.5–2.6].

Theorem (Transfer principle). *Let* M *be a model category cofibrantly generated by the sets* $\mathfrak{I}$ *and* $\mathfrak{J}$ *of generating cofibrations and generating acyclic cofibrations respectively. Let* C *be a category with finite limits and small colimits. Let*

$$L\colon \mathsf{M} \rightleftarrows \mathsf{C} : R$$

be a pair of adjoint functors. Define a map f *in* C *to be a weak equivalence (respectively fibration) if* $R(f)$ *is a weak equivalence (respectively fibration). These two classes determine a model category structure on* C *cofibrantly generated by* $L(\mathfrak{I})$ *and* $L(\mathfrak{J})$ *provided that:*

(1) *The functor* L *preserves small objects;*

(2) C *has a functorial fibrant replacement and a functorial path object for fibrant objects.*

Furthermore, with this model structure on C*, the adjunction* $L \dashv R$ *becomes a Quillen pair.*

We first check that the weak equivalences and fibrations, obtained by transferring the model structure on $\mathsf{pcDGMod}\text{-}\check{B}A$ along the adjunction $G \dashv F$, admit the characterisations in Theorem 5.2.8. In fact, both the functors F and G preserve weak equivalences between all objects.

Lemma 5.2.9.

(1) *A morphism* g *of dg* A*-modules is a weak equivalence if and only if* $F(g)$ *is a weak equivalence.*

(2) *A morphism* f *of pseudocompact* $\check{B}A$*-modules is a weak equivalence if and only if* $G(f)$ *is a weak equivalence.*

Proof. For (1), let $g\colon M \to N$ be a map of dg A-modules. By definition $F(g)\colon FM \to FN$ is a weak equivalence if and only if it induces a quasi-isomorphism

$$\mathrm{Hom}_{\check{B}A}(FM, V) \to \mathrm{Hom}_{\check{B}A}(FN, V)$$

for any finite-dimensional dg $\check{B}A$-module V. Equivalently, this says that the dg A-modules $M \otimes V$ and $N \otimes V$ (with possibly twisted differentials) are quasi-isomorphic for any finite-dimensional V, that is, g is a weak equivalence.

For (2), it suffices to show that G takes exact triples of $\check{B}A$-modules to weakly trivial A-modules. Let $N_1 \to N_2 \to N_3$ be an exact triple of $\check{B}A$-modules and N be its total complex. Then GN is the total complex of the complex $G(N_3) \to G(N_2) \to G(N_1)$, which is a bicomplex with three vertical columns and the all horizontal rows exact.

Now let $M = A \otimes V$ be a finitely generated twisted A-module. Applying $\mathrm{Hom}_A(M, -)$ to the above bicomplex gives $\mathrm{Hom}(V, G(N_3)) \to \mathrm{Hom}(V, G(N_2)) \to \mathrm{Hom}(V, G(N_1))$. Since exactness of the rows is preserved, GN is indeed weakly trivial. □

Lemma 5.2.10. *A morphism g of dg A-modules is a fibration if and only if $F(g)$ is a fibration.*

Proof. Let $g\colon M \to N$ be a fibration in dg A-modules, so $M \cong N \oplus V$ for some graded vector space V. Then $F(g)\colon FN \to FM$ is a cofibration in $\mathsf{pcDGMod}\text{-}\check{B}A$ if and only if it is injective with cofibrant cokernel. But indeed, $F(g)\colon (N^* \otimes \check{B}A)^{[\xi]} \to (M^* \otimes \check{B}A)^{[\xi]}$ has cokernel $(V^* \otimes \check{B}A)^{[\xi]}$, which is cofibrant. □

Proof of Theorem 5.2.8. By Lemma 5.2.9 and Lemma 5.2.10, it suffices to check conditions (1) and (2) in the transfer theorem. Condition (1) holds as G preserves small objects, and every object is fibrant so the first part of (2) trivially holds. Hence it only remains to prove that functorial path objects exist for any A-module. Let I be the standard interval object for dg vector spaces, that is, $I = k \oplus \Sigma^{-1}k \oplus k$ with differential $d(a, b, c) = (da, -db + a - c, dc)$. Then for any A-module M, there is a factorisation

$$M \xrightarrow{e} M \otimes I \xrightarrow{(p_1, p_2)} M \oplus M$$

where $e(a) = (a, 0, a)$ and $p_1(a, b, c) = a$, $p_2(a, b, c) = c$. Clearly (p_1, p_2) is a fibration by Lemma 5.2.10. Since I is acyclic, we have a quasi-isomorphism

$$(M \otimes V^*)^{[x]} \to (M \otimes V^*)^{[x]} \otimes I \cong (M \otimes I \otimes V^*)^{[x]}$$

for any finitely generated twisted A-module $(V \otimes A)^{[x]}$, so e is a weak equivalence. Thus $M \otimes I$ is a functorial path object for M. □

We now show that the adjoint pair (F, G) is a Quillen equivalence.

THEOREM 5.2.11. *Let A be an augmented dg algebra and $\check{B}A$ be its extended bar construction.*

(1) *For any dg A-module M, the counit $GFM \to M$ of the adjunction is a weak equivalence of A-modules.*
(2) *For any pseudocompact $\check{B}A$-module N, the counit $FGN \to N$ of the adjunction is a weak equivalence of pseudocompact $\check{B}A$-modules.*

Thus, the Quillen adjunction $G \dashv F$ is a Quillen anti-equivalence between dg A-modules and pseudocompact $\check{B}A$-modules.

PROOF. For any $\check{B}A$-module N, consider

$$BN := \check{B}A \otimes \Sigma^{-1}\bar{A}^* \otimes N,$$

which is a cofibrant resolution of N. Then the functor $G\colon (\mathsf{pcDGMod}\text{-}\check{B}A)^{\mathsf{op}} \to \mathsf{DGMod}\text{-}A$ can also be written as $\mathrm{Hom}_{\check{B}A}(BN, k)$, and the functor $F\colon \mathsf{DGMod}\text{-}A \to (\mathsf{pcDGMod}\text{-}\check{B}A)^{\mathsf{op}}$ is $(M^* \otimes \check{B}A)^{[\xi]}$.

Now for any A-module M, the $\check{B}A$-module $F(M)$ is cofibrant, so $GF(M)$ is quasi-isomorphic to $\mathrm{Hom}(F(M), k) = M$. Cofibrantly replacing M with a twisted module $M \otimes V$, we obtain that M and $GF(M)$ are weakly equivalent.

Conversely, given a $\check{B}A$-module N, the composition $FG(N)$ is the two-term resolution of N from Proposition 5.1.4, so is weakly equivalent to N. □

REMARK 5.2.12. Note that the homotopy category of the constructed model category on dg A-modules is a compactly generated triangulated category (being anti-equivalent to the category of pseudocompact dg modules over a $\check{B}A$) with compact (small) objects being dg modules that are homotopy equivalent to retracts of finitely generated twisted A-modules. We will denote this homotopy category by $D^{\mathrm{II}}_c(A)$.

Example 5.2.13. Consider the dg algebra $A = k[x]/x^2$ with zero differential and x in degree 1. We have $\widetilde{BA} \cong \widetilde{k[x]}$. If k is algebraically closed then the pseudocompact completion $\widetilde{k[x]}$ of $k[x]$ is the product of completions of $k[x]$ at every maximal ideal of $k[x]$, the latter correspond precisely to elements of k. In other words,

$$\widetilde{BA} \cong \widetilde{k[x]} \cong \prod_{\alpha \in k} (k[[x]])_\alpha$$

(this result, in a more general form, is given in [GG99, Example 1.13]). The derived category $D^{\mathrm{II}}_c(A)$ of A of second kind is anti-equivalent to the derived category (of second kind) of pseudocompact modules over $\prod_{\alpha\in k}(k[[x]])_\alpha$ and thus, is drastically different from the ordinary derived category of A. Note that $\mathrm{MC}(A) = \{ax : a \in k\}$; then the twisted A-modules A^ξ for $\xi \in \mathrm{MC}(A)$ are pairwise weakly inequivalent and form a set of compact generators for $D^{\mathrm{II}}(A)$; it is easy to see that it is not possible to choose a single compact generator.

Example 5.2.14. The derived category of second kind D^{II}_c arises in a number of situations of a geometric origin:

(1) Let M be a smooth manifold and $\mathscr{A}^*(M)$ be its smooth de Rham algebra; here the ground field k is $\mathbb{R}$, the real numbers. The choice of a point in M makes $\mathscr{A}^*(M)$ into an augmented dg algebra. A compact object in $D^{\mathrm{II}}_c(\mathscr{A}^*(M))$ is a cohesive $\mathscr{A}^*(M)$-module of [Blo10] and the subcategory of compact objects is equivalent to the triangulated category of perfect cohomologically locally constant complexes of sheaves on M by [CHL21, Theorem 8.1].
(2) Let M be a smooth affine algebraic variety over a field k of characteristic zero having a base point $\mathrm{Spec}(k) \to M$ and $\mathscr{A}^*(M)$ be its *algebraic* de Rham algebra. Then twisted modules over $\mathscr{A}^*(M)$ correspond to D-modules, i.e. modules over the ring of differential operators on M by [Pos11, Theorem B.2] while compact objects in $D^{\mathrm{II}}_c(\mathscr{A}^*(M))$ correspond essentially to *coherent* D-modules.
(3) Let M be a compact complex manifold and $\mathscr{A}^{0*}(M)$ be the Dolbeault algebra of M that can be viewed as augmented by a choice of a base point in M. Again, a compact object $D^{\mathrm{II}}_c(\mathscr{A}^*(M))$ is a cohesive $\mathscr{A}^{0*}(M)$-module and the subcategory of compact objects is equivalent to the derived

category of sheaves on M with coherent cohomology, [Blo10, Theorem 4.1.3] or [CHL21, Theorem 8.3].

5.2.2. Comparison with other weak equivalences in $\mathsf{DGMod}\text{-}A$. Here, we compare the notion of weak equivalences in our model structure on $\mathsf{DGMod}\text{-}A$ with other notions of a weak equivalence from the literature.

Firstly, we can consider the standard model structure on $\mathsf{DGMod}\text{-}A$ where weak equivalences are quasi-isomorphisms and fibrations are surjections. It is clear that any weak equivalence in our model structure is a quasi-isomorphism, by considering A-modules trivially twisted by the Maurer–Cartan element $x = 0$. It follows that $D^{\mathrm{II}}_c(A)$ contains the ordinary derived category of A as a full subcategory. If A is concentrated in nonpositive degrees (e.g. it is an ordinary algebra), or $\bar{A}$ is concentrated in degrees > 1 (e.g. cohomology algebras of simply-connected topological spaces) then by the degree considerations, $\check{B}A \cong \widehat{T}\Sigma^{-1}\bar{A}^*$, the usual bar construction of A from which it follows that our model structure on A-modules is the ordinary one (i.e. of the first kind). Another situation where we obtain the ordinary model category of the first kind is when the dg algebra A is cofibrant. However, for general A, even with a vanishing differential, we get a different result, cf. Example 5.2.13.

In [Pos11], the coderived category and contraderived category of a dg algebra A are defined, which are obtained by localising at coacyclic dg A-modules and contraacyclic dg A-modules respectively. These categories are different, in general, from the ordinary derived category of the first kind, even for ungraded algebras, see e.g. [Pos11, Example 3.3] and thus, also from $D^{\mathrm{II}}_c(A)$.

It was observed in [Pos11, KLN10] that the category $D^{\mathrm{II}}_c(A)$ is contained in both the coderived and contraderived category of A. It is, therefore, the derived category of A of the second kind that is closest to the ordinary derived category of A. If A is right Noetherian and has finite right homological dimension then $D^{\mathrm{II}}_c(A)$ coincides with both coderived and contraderived category of A by [Pos11, Question 3.8]. Another situation when this happens is when A is the cobar construction of a (possibly nonconilpotent) dg coalgebra B since in this case the co/contraderived category of A is equivalent to the coderived category of B and is, therefore, compactly generated. Related questions are considered in the recent paper [Pos17].

5.3. Curved extended Koszul duality for modules

In this section, we consider generalisations of the previous results in the cases where the underlying dg algebra is curved or non-augmented. First we need to develop the extended bar-cobar formalism in the curved, non-augmented context.

A *curved dg algebra* is a graded algebra A with a degree one derivation $d\colon A \to A$, such that for any $a \in A$, $d^2(a) = [h, a]$ for some $h \in A^2$ satisfying $d(h) = 0$. The linear map d is usually called the *differential* of A, despite not being square zero, and h is called the *curvature* of A.

A morphism of curved algebras $(A, d_A, h_A) \to (B, d_B, h_B)$ is a pair (f, b) consisting of a morphism of graded algebras $f\colon A \to B$ and an element $b \in B^1$ satisfying the equations:

$$f(d_A(x)) = d_B(f(x)) + [b, f(x)],$$
$$f(h_A) = h_B + d_B(b) + b^2,$$

for all $x \in A$; if $b = 0$ then the corresponding morphism $A \to B$ is called *strict*. The category of curved dg algebras is denoted by CDG and the category of pseudocompact curved dg algebras is denoted by pcCDG; additionally we assume that our (pseudocompact or not) curved dg algebras have nonzero units. A *Maurer–Cartan element* in a curved dg algebra A is an element $x \in A$ of degree 1 such that $h + dx + x^2 = 0$. Given two curved dg algebras (A, d_A, h_A) and (B, d_B, h_B) their tensor product $A \otimes B$ is likewise a curved dg algebra with $d_{A\otimes B} := d_A \otimes 1 + 1 \otimes d_B$ and $h_{A\otimes B} := h_A \otimes 1 + 1 \otimes h_B$.

Given a curved dg algebra (A, d_A, h_A) and an element $b \in A^1$ (not necessarily Maurer–Cartan) we can define the *twisting* of A by b as a curved dg algebra A^b with the same underlying vector space as A, twisted differential $d^b(x) := d_A(x) + [b, x]$ for $x \in A$ and curvature $h^b := h_A + d_A(b) + b^2$. Then (id, b) determines a (curved) isomorphism $A^b \to A$.

If A is a curved dg algebra, then a *dg A-module* is a graded (right) A-module M with a degree one derivation $d_M\colon M \to M$ such that d_M is compatible with the differential d on A, and for any $m \in M$, $d_M^2(m) = mh$; one can similarly define left dg A-modules. If M is a left dg A-module and $x \in A^1$, then there is a left dg A^x-module $M^{[x]}$ defined as in the uncurved case, cf. Definition 5.2.1. Given a curved dg algebra A and a pseudocompact curved dg algebra C, we denote the categories of dg A-modules and pseudocompact C-modules by $\mathsf{DGMod}\text{-}A$ and $\mathsf{pcDGMod}\text{-}C$, just as before.

We now describe how to modify the bar and cobar constructions from Definition 5.1.5 in the general non-augmented and curved case. Let A be a unital curved dg algebra with differential d and curvature h. Since $1 \neq 0$ in A we can choose a homogeneous k-linear retraction $\epsilon \colon A \to k$, to be regarded as a "fake augmentation". It allows one to identify the dg vector space $\bar{A} := A/k$ with a subspace (possibly not dg) of A so that $A \cong k \oplus \bar{A}$. The multiplication $m \colon A \otimes A \to A$ restricted to $\bar{A}$ has two components $m^{\epsilon}_{\bar{A}} \colon \bar{A} \otimes \bar{A} \to \bar{A}$ and $m^{\epsilon}_{k} \colon \bar{A} \otimes \bar{A} \to k$. We will denote the corresponding components of the differential d and curvature h by $d^{\epsilon}_{\bar{A}}, d^{\epsilon}_{k}$ and $h^{\epsilon}_{\bar{A}}$, respectively; note that the component h^{ϵ}_{k} vanishes for degree reasons. Explicitly, for all $\bar{a}, \bar{b} \in \bar{A} \subset A$,

$$\begin{aligned} &m^{\epsilon}_{\bar{A}}(\bar{a}, \bar{b}) = \bar{a}\bar{b} - \epsilon(\bar{a}\bar{b}),\ \ m^{\epsilon}_{k}(\bar{a}, \bar{b}) = \epsilon(\bar{a}\bar{b}); \\ &d^{\epsilon}_{\bar{A}}(\bar{a}) = d(\bar{a}) - \epsilon(d(\bar{a})),\ \ d^{\epsilon}_{k}(\bar{a}) = \epsilon(d(\bar{a})); \\ &h^{\epsilon}_{\bar{A}} = h - \epsilon(h) = h. \end{aligned} \tag{5.3.1}$$

To alleviate notation, we will suppress the superscript ϵ at $m_{\bar{A}}$, m_k etc. where it does not lead to confusion.

Consider the graded algebra $T'\Sigma^{-1}A^*$, the non-reduced semi-completed bar construction of A. Choose a basis $\{t^i : i \in I\}$ in $\bar{A}$ where I is some indexing set and let $\{\tau, t_i : i \in I\}$ be the basis in $\Sigma^{-1}A^*$ dual to the basis $\{1, t_i : i \in I\}$ in A. We will write ∂_{t_i} for the derivation of $T'\Sigma^{-1}A^*$ having value 1 on t_i and zero on other basis elements of $\Sigma^{-1}A^*$ and similarly for ∂_τ. Then define the differential on $T'\Sigma^{-1}A^*$ as the following derivation:

$$\xi := \sum_{i\in I}([\tau, t_i] + f_i(\mathbf{t}))\partial_{t_i} + (g(\mathbf{t}) + \tau^2)\partial_\tau + \sum_{i\in I} a_i\partial_{t_i}$$

where $f_i(\mathbf{t})$, $g(\mathbf{t})$ stand for sums of linear and quadratic monomials in t (so these elements of $T'\Sigma^{-1}A^*$ do not depend on τ). Here the term $\sum_{i\in I} f_i(\mathbf{t})\partial_{t_i}$ corresponds to the "multiplication and differential" $m_{\bar{A}}$ and $d_{\bar{A}}$, the term $\sum_{i\in I} a_i\partial_{t_i}$ reflects the curvature $h_{\bar{A}}$, the term $g(\mathbf{t})\partial_\tau$ corresponds to m_k and d_k, and the term $(\sum_{i\in I}[\tau, t_i] + \tau^2)\partial_\tau$ reflects the multiplication with the unit in A. Let $\xi_1 := \sum_{i\in I} f_i(\mathbf{t})\partial_{t_i} + \sum_{i\in I} a_i\partial_{t_i}$ and $\xi_2 := \sum_{i\in I}[\tau, t_i]\partial_{t_i} + (g(\mathbf{t}) + \tau^2)\partial_\tau$; then $\xi = \xi_1 + \xi_2$.

The *reduced* semi-complete bar construction $B'_\epsilon A$ of A is a subalgebra in $T'\Sigma^{-1}A^*$ spanned by sums of monomials which do not depend on τ (so only depend on t_i,

$i \in I$). Thus, the underlying graded algebra of $B'_\epsilon A$ is isomorphic to $T'\Sigma^{-1}\bar{A}^*$. The differential on $B'_\epsilon A$ is ξ_1. Note that $\xi^2 = 0$ but $\xi_1^2 = 0$ only when $\epsilon\colon A \to k$ is a dg algebra map; in this case $g(\mathbf{t}) = 0$. However $(B'_\epsilon A, \xi_1)$ is a *curved* dg algebra, more precisely the following result holds.

LEMMA 5.3.1. *Let A be a curved dg algebra. Then:*

(1) *The reduced semi-complete bar construction $B'_\epsilon A$ endowed with the differential ξ_1 defined above, is a curved dg algebra with curvature $-g(-\mathbf{t})$, an element of $T'\Sigma^{-1}\bar{A}^*$ obtained from $-g(\mathbf{t})$ by replacing every indeterminate t_i with $-t_i$.*

(2) *The curved dg algebra $B'_\epsilon A$ is independent, up to a natural isomorphism, of the choice of a basis in $\bar{A}$. Furthermore, for different choices of retractions $A \to k$, the corresponding reduced semi-complete bar constructions are isomorphic as curved dg algebras. More precisely, denote by $b_{\epsilon-\epsilon'}$ the element in $B'A \cong T'\Sigma^{-1}\bar{A}^*$ corresponding to the linear map $\epsilon - \epsilon'\colon A \to k$; then the curved map $(\mathrm{id}, b_{\epsilon-\epsilon'})$ determines a curved isomorphism $B'_\epsilon A \to B'_{\epsilon'} A$.*

(3) *The correspondence $A \to B'_\epsilon A$ determines a contravariant functor from the category CDG to the category of topological curved dg algebras.*

PROOF. Taking into account that $0 = \xi^2 = \xi_1^2 + [\xi_1, \xi_2] + \xi_2^2$ we have for $k \in I$,

$$\xi_1^2(t_k) = -[\xi_1, \xi_2](t_k) - \xi_2^2(t_k).$$

Furthermore, a straightforward calculation shows that $[\xi_1, \xi_2](t_k)$ has no terms depending on t_i, $i \in I$ whereas the only term of $\xi_2^2(t_k)$ depending on t_i, $i \in I$ has the form $g(\mathbf{t})\partial_\tau([t_k, \tau]) = (-1)^{|t_k|}[t_k, g(\mathbf{t})]$. It follows that

$$\xi_1^2(t_k) = -(-1)^{|t_k|}[g(\mathbf{t}), t_k]$$

as required.

Next, the statement about the independence of $B'_\epsilon(A)$ on a basis in $\bar{A}$ is obvious. Let $\epsilon'\colon A \to k$ be another fake augmentation; then formulas (5.3.1) show that h is unchanged whereas $m_{\bar{A}}^{\epsilon'}(\bar{a}, \bar{b}) = m_{\bar{A}}^{\epsilon}(\bar{a}, \bar{b}) + (\epsilon - \epsilon')(\bar{a}\bar{b})$, and similarly for the differential. This implies that $B'_{\epsilon'} A$ is obtained from $B'_\epsilon A$ by twisting with the element $\epsilon - \epsilon' \in B'_\epsilon A$, which is equivalent to the stated claim.

To see that the construction $A \to B'_\epsilon A$ is functorial, we will view an object in CDG as a curved dg algebra A *with a choice of a retraction* $A \to k$, however morphisms need not respect the retraction; this is clearly the same as (or, more accurately, equivalent to) the category CDG. Any map $A \to B$ in CDG can canonically be factorized in CDG as $A \to A \to B$ with the first map being a change of retraction in A followed by a map preserving retractions. The construction $B'_\epsilon A$ is clearly functorial with respect to retraction-preserving maps and a change of retraction is also functorial by part (2). □

This allows us to define the extended bar construction of a curved dg algebra in the same way as it was done in the uncurved case; from now on we will suppress the subscript ϵ and write $B'_\epsilon A$ for the semi-complete bar construction of A; by Lemma 5.3.1 this specifies a curved pseudocompact dg algebra up to a natural isomorphism.

Definition 5.3.2. Let A be a curved dg algebra with a retraction $\epsilon\colon A \to k$. The *extended bar construction* of A is the graded pseudocompact algebra

$$\check{B}A := \check{T}\Sigma^{-1}\bar{A}^*.$$

Then by Proposition 5.1.2(1), the identity on $\check{T}\Sigma^{-1}\bar{A}^*$ induces a map $i\colon B'A \cong T'(\Sigma^{-1}\bar{A}^*) \to \check{B}A \cong \check{T}(\Sigma^{-1}\bar{A}^*)$, and we define the differential $d_{\check{B}A}$ on $\check{B}A$ to be

$$d_{\check{B}A} := i \circ \xi_1\colon \Sigma^{-1}\bar{A}^* \to T'(\Sigma^{-1}\bar{A}^*) \to \check{T}(\Sigma^{-1}\bar{A}^*).$$

The curvature of $\check{B}A$ is the image of the curvature in $B'A$ under the map $i\colon B'A \to \check{B}A$. This gives $\check{B}A$ the structure of a curved pseudocompact dg algebra.

Remark 5.3.3. It follows from Lemma 5.3.1 that the correspondence $A \mapsto \check{B}A$ is a functor $\mathsf{CDG} \to \mathsf{pcCDG}^{\mathsf{op}}$. A version of the definition above with $\widehat{T}\Sigma^{-1}\bar{A}^*$ (the *local* pseudocompact bar construction of a curved non-augmented algebra) in place of $\check{T}\Sigma^{-1}\bar{A}^*$ is found in [Pos11, Section 6.1], albeit formulated in the language of coalgebras. However Positselski's local bar construction is *not* functorial with respect to non-strict maps in CDG since maps between pseudocompact algebras of the form $\widehat{T}\Sigma^{-1}\bar{A}^*$ must preserve their maximal ideals whereas this is not true for pseudocompact algebras of the form $\check{T}\Sigma^{-1}\bar{A}^*$ (which can have many maximal ideals).

Now recall that given a pseudocompact curved dg algebra C there is defined a curved dg algebra

$$\Omega C := T\Sigma^{-1}\bar{C}^*$$

with $\bar{C} := C/k$, cf. [Pos11, Section 6.1]. Note that the definition of Ω can be given along the lines of the definition of $\check{B}$, only simpler since there is no analogue, or need, for an intermediate step involving the semi-complete bar construction. Then we have the following result.

PROPOSITION 5.3.4. *The correspondence $C \mapsto \Omega(C)$ determines a functor $\mathsf{pcCDG}^{\mathsf{op}} \to \mathsf{CDG}$. This functor is left adjoint to $\check{B}\colon \mathsf{CDG} \to \mathsf{pcCDG}^{\mathsf{op}}$.*

PROOF. The functoriality of ΩC was explained in [Pos11, Section 6.1]; alternatively the arguments in the proof of Lemma 5.3.1 apply with obvious modifications. The adjointness follows as in the non-curved case; namely by noticing that for $A \in \mathsf{CDG}$, $C \in \mathsf{pcCDG}$ the sets of morphisms $\mathrm{Hom}_{\mathsf{CDG}}(\Omega C, A)$ and $\mathrm{Hom}_{\mathsf{pcCDG}}(\check{B}A, C)$ are both naturally isomorphic to $\mathrm{MC}(A \otimes C)$. □

REMARK 5.3.5. If a curved dg algebra A is happens to be augmented, then there is a natural choice of a retraction $\epsilon\colon A \to k$, namely, the given augmentation. In this case, $\check{B}A$ is uncurved. Similarly, if A has vanishing curvature, $\check{B}A$ is naturally augmented. If A is both augmented and uncurved, then so is $\check{B}A$.

Now for a curved dg algebra A and its bar construction $\check{B}A$, there is an adjunction

$$G\colon \mathsf{pcDGMod}\text{-}\check{B}A^{\mathsf{op}} \rightleftarrows \mathsf{DGMod}\text{-}A : F \tag{5.3.2}$$

as defined in Definition 5.2.3; these functors are well-defined as the twisting of a curved dg algebra by a Maurer–Cartan element gives an uncurved dg algebra. Furthermore, Theorem 5.2.6 holds (with the same definitions of weak equivalences, fibrations and cofibrations) when the dg coalgebra C is curved (indeed, this is how it was formulated in [Pos11]). Thus, $\mathsf{pcDGMod}\text{-}\check{B}A^{\mathsf{op}}$ has the structure of a model category and by transferring along the adjunction (5.3.2) we obtain the following generalisation of Theorem 5.2.8; the arguments are the same as in the uncurved case.

THEOREM 5.3.6. *Let A be a curved dg algebra. There is a cofibrantly generated model category structure on $\mathsf{DGMod}\text{-}A$, where*

(1) *a morphism* $f\colon M \to N$ *is a* weak equivalence *if it induces a quasi-isomorphism*

$$\mathrm{Hom}_A((V\otimes A)^{[x]}, M) \to \mathrm{Hom}_A((V\otimes A)^{[x]}, N)$$

for any finitely generated twisted A*-module* $(V\otimes A)^{[x]}$*;*

(2) *a morphism is a* fibration *if it is surjective;*

(3) *a morphism is a* cofibration *if it has the left lifting property with respect to acyclic fibrations.*

With this model structure, the adjunction $G \dashv F$ *is a Quillen pair.*

Similarly, there are model structures on $\mathsf{DGMod}\text{-}A$ when A is curved and augmented, or non-curved and non-augmented. Altogether there are four cases as below. Case (4) is the case considered previously and proved in Theorem 5.2.11. Again, the arguments employed in the augmented uncurved case generalize in a straightforward fashion.

THEOREM 5.3.7. *With the above model structures, the functors* $G \dashv F$ *form a Quillen anti-equivalence between the categories* $\mathsf{pcDGMod}\text{-}\check{B}A$ *and* $\mathsf{DGMod}\text{-}A$ *in each of the following four cases:*

(1) A *is curved and non-augmented,* $\check{B}A$ *is curved and non-augmented;*
(2) A *is curved and augmented,* $\check{B}A$ *is non-curved and non-augmented;*
(3) A *is non-curved and non-augmented,* $\check{B}A$ *is curved and augmented;*
(4) A *is non-curved and augmented,* $\check{B}A$ *is non-curved and augmented.*

Notation

A list of notation for the various categories we use is included here as a reference tool. For more details, refer to the page number in the rightmost column.

Vect	$\mathbb{Z}$-graded vector spaces	17
DGVect	differential graded (dg) vector spaces	18
Alg	$\mathbb{Z}$-graded algebras	17
CAlg	$\mathbb{Z}$-graded commutative algebras	17
DGA	dg algebras	19
DGA^*	augmented dg algebras	19
CDGA	commutative dg algebras	19
pcX	any category X above with pseudocompact topology	20
DGLA	dg Lie algebras	19
$\mathsf{pcDGA}_{\mathsf{loc}}$	local augmented pseudocompact dg algebras	23
$\mathsf{pcCDGA}_{\mathsf{loc}}$	local augmented pseudocompact commutative dg algebras	23
DGMod-A	right dg modules over algebra A	19
pcDGMod-A	right dg modules over a pseudocompact algebra A	23
CDG	curved dg algebras (Chapter 5 only)	84
pcCDG	curved pseudocompact dg algebras (Chapter 5 only)	84

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